Appendix F

Transport Equations used in STREAM

The STREAM code uses a nonorthogonal curvilinear grid arrangement but aligns velocity components to a Cartesian reference frame. The following sections briefly describe the derivation of the RANS equations in this coordinate system (see also Lien [73]).

F.1 Introduction to Hybrid Curvilinear-Cartesian Coordinates

F.1.1 Vector Components

The velocity vector, $\mathbf{v}$, can be represented as either a function of the covariant base vectors, $\mathbf{g}_j$, or as a function of the Cartesian base vectors, $\mathbf{e}_m$, as follows:

$$\mathbf{v} = v^j \mathbf{g}_j = u^m \mathbf{e}_m \tag{F.1}$$

This can be rearranged:

$$v^j = u^m \mathbf{e}_m \cdot \mathbf{g}^j = u^m \beta^j_m \tag{F.2}$$

where, from Section B.3:

$$\beta^i_k = \mathbf{e}_k \cdot \mathbf{g}^i = \frac{\partial \xi^j}{\partial x^k} \mathbf{g}_j \cdot \mathbf{g}^i = \frac{\partial \xi^j}{\partial x^k} \delta^i_j = \underbrace{\frac{\partial \xi^i}{\partial x^k}}_{[J]^{-1}} \tag{F.3}$$

The term δ^i_j is the Kronecker delta (which has substitution-operator properties) and $\partial \xi^j / \partial x^m$ are elements of the inverse Jacobian matrix, $[J]^{-1}$.

F.1.2 Covariant Derivative of Vector, v

The derivative of the Cartesian vector in Cartesian coordinates is expanded:

$$\frac{\partial}{\partial x^j}\left(u^i \mathbf{e}_i\right) = \frac{\partial u^i}{\partial x^j}\mathbf{e}_i + \underbrace{u^i \frac{\partial \mathbf{e}_i}{\partial x^j}} \tag{F.4}$$

where the underbraced term is zero since the Cartesian base vectors do not vary with position. Using the chain-rule this can also be expressed in terms of the derivative in the covariant coordinates:

$$\frac{\partial}{\partial x^j}\left(u^i \mathbf{e}_i\right) = \frac{\partial u^i}{\partial \xi^k}\frac{\partial \xi^k}{\partial x^j}\mathbf{e}_i = \frac{\partial u^i}{\partial \xi^k}\beta^k_j \mathbf{e}_i \tag{F.5}$$

F.1.3 Covariant Derivative of Tensor, T

The Cartesian derivative of the second-order tensor, $\mathbf{T} = \tau^{ij}\mathbf{e}_i \otimes \mathbf{e}_j$, is given by:

$$\begin{aligned}\frac{\partial \mathbf{T}}{\partial x^k} &= \frac{\partial}{\partial x^k}\left(\tau^{ij}\mathbf{e}_i \otimes \mathbf{e}_j\right) \\ &= \frac{\partial \tau^{ij}}{\partial x^k}\mathbf{e}_i \otimes \mathbf{e}_j \end{aligned} \tag{F.6}$$

This can be expressed in terms of the curvilinear components, ξ^m, using the chain-rule, as follows:

$$\begin{aligned}\frac{\partial \mathbf{T}}{\partial x^k} &= \frac{\partial \tau^{ij}}{\partial \xi^m}\frac{\partial \xi^m}{\partial x^k}\mathbf{e}_i \otimes \mathbf{e}_j \\ &= \frac{\partial \tau^{ij}}{\partial \xi^m}\beta^m_k \mathbf{e}_i \otimes \mathbf{e}_j \end{aligned} \tag{F.7}$$

F.1.4 Gradient of a Scalar, $\nabla\phi$

The gradient of the scalar, ϕ, can be expanded as follows:

$$\begin{aligned}\nabla\phi &= \frac{\partial \phi}{\partial x^j}\mathbf{e}^j \\ &= \frac{\partial \phi}{\partial \xi^j}\frac{\partial \xi^k}{\partial x^j}\mathbf{e}^j \\ &= \frac{\partial \phi}{\partial \xi^j}\beta^k_j \mathbf{e}_j \end{aligned} \tag{F.8}$$

where the covariant and contravariant Cartesian base vectors are equivalent $\mathbf{e}_j \equiv \mathbf{e}^j$.

F.1.5 Gradient of a Vector, $\nabla \mathbf{v}$

The gradient of the velocity vector in Cartesian coordinates is given by:

$$\nabla \mathbf{v} = \mathbf{e}^j \frac{\partial}{\partial x^j} \left(u^i \mathbf{e}_i \right) \tag{F.9}$$

In Equation (F.5) it was shown that the covariant derivative of a Cartesian vector is as follows:

$$\frac{\partial}{\partial x^j} \left(u^i \mathbf{e}_i \right) = \frac{\partial u^i}{\partial \xi^k} \beta^k_j \mathbf{e}_i \tag{F.10}$$

The gradient of $\mathbf{v}$ therefore becomes:

$$\nabla \mathbf{v} \;\; = \;\; \frac{\partial u^i}{\partial \xi^k} \beta^k_j \mathbf{e}_i \otimes \mathbf{e}_j \tag{F.11}$$

The components of the tensor $\nabla \mathbf{v} = \tau^{ij} \mathbf{e}_i \otimes \mathbf{e}_j$ are therefore:

$$\tau^{ij} = \beta^k_j \frac{\partial u^i}{\partial \xi^k} \tag{F.12}$$

F.1.6 Divergence of a Vector, $\nabla \cdot \mathbf{v}$

The divergence of vector $\mathbf{v} = u^i \mathbf{e}_i$ is given by:

$$\nabla \cdot \mathbf{v} = \mathbf{e}^k \cdot \frac{\partial}{\partial x^k} \left(u^i \mathbf{e}_i \right) \tag{F.13}$$

Substituting Equation (F.5) into the above expression and rearranging:

$$\begin{aligned} \nabla \cdot \mathbf{v} \;\; &= \;\; \frac{\partial u^i}{\partial \xi^j} \beta^j_k \mathbf{e}_i \cdot \mathbf{e}^k \\ &= \;\; \frac{\partial u^i}{\partial \xi^j} \beta^j_k \delta^k_i \\ &= \;\; \frac{\partial u^i}{\partial \xi^j} \beta^j_i \end{aligned} \tag{F.14}$$

An alternative expression for $\nabla \cdot \mathbf{v}$ can be obtained by considering the divergence of vector $\mathbf{v}$ in curvilinear coordinates, as given by Equation (B.133):

$$\nabla \cdot \mathbf{v} = \frac{1}{J} \frac{\partial}{\partial \xi^j} \left(J v^j \right) \tag{F.15}$$

Converting the curvilinear component v^j into Cartesian components, using Equation (F.2), one obtains:

$$\nabla \cdot \mathbf{v} = \frac{1}{J} \frac{\partial}{\partial \xi^j} \left(J u^i \beta^j_i \right) \tag{F.16}$$

Equations (F.14) and (F.16) are equivalent, therefore one can write:

$$\frac{\partial u^i}{\partial \xi^j}\beta_i^j = \frac{1}{J}\frac{\partial}{\partial \xi^j}\left(Ju^i\beta_i^j\right) \tag{F.17}$$

Expanding the right-hand-side of the above equation using the product rule, one obtains:

$$\frac{\partial u^i}{\partial \xi^j}\beta_i^j = \frac{\partial u^i}{\partial \xi^j}\beta_i^j + \underbrace{\frac{u^i}{J}\frac{\partial}{\partial \xi^j}\left(J\beta_i^j\right)}_{zero} \tag{F.18}$$

where, in order for the two sides of the above equation to be equal, the underbraced term must be zero.

F.1.7 Divergence of a Tensor, $\nabla \cdot \mathbf{T}$

Divergence of the second-order tensor, $\mathbf{T} = \tau^{ij}\mathbf{e}_i \otimes \mathbf{e}_j$ in Cartesian coordinates is given by:

$$\nabla \cdot \mathbf{T} = \mathbf{e}^k \cdot \frac{\partial}{\partial x^k}\left(\tau^{ij}\mathbf{e}_i \otimes \mathbf{e}_j\right) \tag{F.19}$$

This can also be written, using Equation (F.7):

$$\begin{aligned}
\nabla \cdot \mathbf{T} &= \mathbf{e}^k \cdot \frac{\partial \tau^{ij}}{\partial \xi^m}\beta_k^m \mathbf{e}_i \otimes \mathbf{e}_j \\
&= \frac{\partial \tau^{ij}}{\partial \xi^m}\beta_k^m \delta_j^k \mathbf{e}_i \\
&= \frac{\partial \tau^{ij}}{\partial \xi^m}\beta_j^m \mathbf{e}_i
\end{aligned} \tag{F.20}$$

F.1.8 Summary of Transformation Rules

The following equations are used to derive the RANS equations in a coordinate system where derivatives are taken with respect to non-orthogonal curvilinear vectors $\mathbf{r} = \xi^i \mathbf{g}_i$, whilst vector and tensor parameters are aligned to a Cartesian coordinate system ($\mathbf{v} = u^i\mathbf{e}_i$ and $\mathbf{T} = \tau^{ij}\mathbf{e}_i \otimes \mathbf{e}_j$).

Scalars

$$\nabla \phi \rightarrow \frac{\partial \phi}{\partial \xi^j}\beta_i^j \mathbf{e}_i \tag{F.21}$$

Vectors

$$\nabla \mathbf{v} \rightarrow \beta_j^k \frac{\partial u^i}{\partial \xi^k}\mathbf{e}_i \otimes \mathbf{e}_j \tag{F.22}$$

$$\nabla \cdot \mathbf{v} \rightarrow \frac{\partial u^i}{\partial \xi^k}\beta_i^k \equiv \frac{1}{J}\frac{\partial}{\partial \xi^j}\left(J u^m \beta_m^j\right) \tag{F.23}$$

Tensors

$$\nabla \cdot \mathbf{T} \rightarrow \frac{\partial \tau^{ij}}{\partial \xi^m}\beta_j^m \mathbf{e}_i \tag{F.24}$$

F.2 Transport Equations in Hybrid Coordinates

F.2.1 Scalar

The transport equation for scalar, ϕ, in vector form is given by:

$$\nabla \cdot (\rho \mathbf{U}\phi - \mathbf{q}) = S_\phi \tag{F.25}$$

where the scalar flux vector is given by:

$$\mathbf{q} = \Gamma_\phi \nabla \phi \tag{F.26}$$

where Γ_ϕ is the diffusivity. Using the equations given in Section F.1.8, one obtains the following scalar transport equation:

$$\frac{\partial}{\partial \xi^j}\left(J\beta_m^j \rho U^m \phi - J\beta_m^j \beta_m^n \Gamma_\phi \frac{\partial \phi}{\partial \xi^n}\right) = J S_\phi \tag{F.27}$$

where the velocity components, U^m, are aligned to Cartesian base vectors and gradients are taken with respect to the curvilinear components, ξ^j. The term J is the Jacobian which is equivalent to the cell volume (see Section B.5).

F.2.2 Momentum

The Reynolds-averaged momentum equation can be written in vector form:

$$\nabla \cdot (\rho \mathbf{U} \otimes \mathbf{U} - \mathbf{T}) = -\nabla P \tag{F.28}$$

where $\mathbf{U}$ is the mean velocity vector, $\mathbf{T}$ is the second-order stress tensor and P is the mean pressure. The pressure gradient term is expanded:

$$-\nabla P = -\frac{\partial P}{\partial \xi^j}\beta_i^j \mathbf{e}_i \tag{F.29}$$

which can also be written, using Equation (F.18) and the product rule:

$$\begin{aligned} -\nabla P &= -\left[\frac{\partial P}{\partial \xi^j}\beta_i^j + \frac{P}{J}\frac{\partial}{\partial \xi^j}\left(J\beta_i^j\right)\right]\mathbf{e}_i \\ &= -\frac{1}{J}\frac{\partial}{\partial \xi^j}\left(J\beta_i^j P\right)\mathbf{e}_i \end{aligned} \quad \text{(F.30)}$$

The divergence of the convective flux is expanded:

$$\nabla \cdot (\rho \mathbf{U} \otimes \mathbf{U}) = \frac{\partial \left(\rho U^i U^m\right)}{\partial \xi^j}\beta_m^j \mathbf{e}_i \quad \text{(F.31)}$$

Following the same approach used above for the pressure gradient, Equation (F.31) can be written:

$$\nabla \cdot (\rho \mathbf{U} \otimes \mathbf{U}) = \frac{1}{J}\frac{\partial}{\partial \xi^j}\left(J\beta_m^j \rho U^i U^m\right)\mathbf{e}_i \quad \text{(F.32)}$$

The stress tensor is given by:

$$\mathbf{T} = \tau^{im}\mathbf{e}_i \otimes \mathbf{e}_m \quad \text{(F.33)}$$

with Cartesian components:

$$\tau^{im} = \mu\left(\frac{\partial U^i}{\partial x^m} + \frac{\partial U^m}{\partial x^i}\right) + \delta^{im}\lambda\frac{\partial U^k}{\partial x^k} - \rho\overline{u^i u^m} \quad \text{(F.34)}$$

where $\lambda = -2\mu/3$ is the bulk viscosity. The above equation can be rearranged, using the chain-rule to express gradients in curvilinear coordinates, as follows:

$$\tau^{im} = \mu\left(\frac{\partial U^i}{\partial \xi^n}\beta_m^n + \frac{\partial U^m}{\partial \xi^n}\beta_i^n\right) + \delta^{im}\lambda\frac{\partial U^k}{\partial \xi^n}\beta_k^n - \rho\overline{u^i u^m} \quad \text{(F.35)}$$

The divergence of the stress tensor is then:

$$\begin{aligned} \nabla \cdot \mathbf{T} &= \frac{1}{J}\frac{\partial}{\partial \xi^j}\left(J\beta_m^j \tau^{im}\right)\mathbf{e}_i \\ &= \frac{1}{J}\frac{\partial}{\partial \xi^j}\left[J\beta_m^j \mu\left(\frac{\partial U^i}{\partial \xi^n}\beta_m^n + \frac{\partial U^m}{\partial \xi^n}\beta_i^n\right) + \delta^{im}\lambda\frac{\partial U^k}{\partial \xi^n}\beta_k^n - J\beta_m^j \rho\overline{u^i u^m}\right]\mathbf{e}_i \end{aligned} \quad \text{(F.36)}$$

Finally, combining Equations (F.30), (F.32) and (F.36), one obtains the following expression for the momentum equation in hybrid Cartesian-curvilinear coordinates:

$$\begin{aligned} &\frac{\partial}{\partial \xi^j}\left(J\beta_m^j \rho U^i U^m - J\beta_m^j \beta_m^n \mu\frac{\partial U^i}{\partial \xi^n}\right) \\ &= \frac{\partial}{\partial \xi^j}\left(-J\beta_i^j P + J\beta_m^j \beta_i^n \mu\frac{\partial U^m}{\partial \xi^n} + J\beta_m^j \beta_k^n \delta^{im}\lambda\frac{\partial U^k}{\partial \xi^n} - J\beta_m^j \rho\overline{u^i u^m}\right) \end{aligned} \quad \text{(F.37)}$$