Appendix A

RANS Equations for Axisymmetric Swirling Flow

The RANS equations for steady, incompressible flow are presented below in cylindrical-polar coordinates, using a stationary reference frame. The velocity components in the radial (r), axial (y) and tangential (ϕ) directions are denoted U, V and W respectively. Convection terms are shown in conservative form. For confirmation of these equations see Owen & Wilson [112] or Morse [124].

Continuity

$$\frac{1}{r}\frac{\partial}{\partial r}(rU) + \frac{\partial V}{\partial y} = 0 \tag{A.1}$$

Radial Momentum, U

$$\begin{aligned}
\frac{\partial}{\partial r}(\rho r U U) + \frac{\partial}{\partial y}(\rho r U V) - \rho W^2 \quad = \quad & -r\frac{\partial P}{\partial r} + \frac{\partial}{\partial r}\left(2r\mu\frac{\partial U}{\partial r} - \rho r\overline{u^2}\right) \\
& + \frac{\partial}{\partial y}\left[r\mu\left(\frac{\partial U}{\partial y} + \frac{\partial V}{\partial r}\right) - \rho r\overline{uv}\right] \\
& - \left(2\mu\frac{U}{r} - \rho\overline{w^2}\right)
\end{aligned} \tag{A.2}$$

Axial Momentum, V

$$\begin{aligned}
\frac{\partial}{\partial r}(\rho r U V) + \frac{\partial}{\partial y}(\rho r V V) \quad = \quad & -r\frac{\partial P}{\partial y} + \frac{\partial}{\partial r}\left[r\mu\left(\frac{\partial V}{\partial r} + \frac{\partial U}{\partial y}\right) - \rho r\overline{uv}\right] \\
& + \frac{\partial}{\partial y}\left(2r\mu\frac{\partial V}{\partial y} - \rho r\overline{v^2}\right)
\end{aligned} \tag{A.3}$$

Tangential Momentum, W

$$\begin{aligned}\frac{\partial}{\partial r}(\rho r U W)+\frac{\partial}{\partial y}(\rho r V W)+\rho U W &= \frac{\partial}{\partial r}\left(r\mu\frac{\partial W}{\partial r}-\rho r\overline{uw}\right)+\frac{\partial}{\partial y}\left(r\mu\frac{\partial W}{\partial y}-\rho r\overline{vw}\right)\\ &\quad -\left(\frac{\mu W}{r}+W\frac{\partial\mu}{\partial r}+\rho\overline{uw}\right) \qquad \text{(A.4)}\end{aligned}$$

A.1 Linear $k-\varepsilon$ Model

Reynolds Stress, $\overline{u_i u_j}$

In a linear $k-\varepsilon$ model the Reynolds stresses, $\overline{u_i u_j}$, are as follows:

$$-\rho\overline{u^2}=2\mu_t\frac{\partial U}{\partial r}-\frac{2}{3}\rho k \qquad \text{(A.5)}$$

$$-\rho\overline{v^2}=2\mu_t\frac{\partial V}{\partial y}-\frac{2}{3}\rho k \qquad \text{(A.6)}$$

$$-\rho\overline{w^2}=2\mu_t\frac{U}{r}-\frac{2}{3}\rho k \qquad \text{(A.7)}$$

$$-\rho\overline{uv}=\mu_t\left(\frac{\partial U}{\partial y}+\frac{\partial V}{\partial r}\right) \qquad \text{(A.8)}$$

$$-\rho\overline{uw}=\mu_t r\frac{\partial}{\partial r}\left(\frac{W}{r}\right)=\mu_t\left(\frac{\partial W}{\partial r}-\frac{W}{r}\right) \qquad \text{(A.9)}$$

$$-\rho\overline{vw}=\mu_t\frac{\partial W}{\partial y} \qquad \text{(A.10)}$$

Substituting these into the above RANS equations:

Radial Momentum, U

$$\begin{aligned}\frac{\partial}{\partial r}(\rho r U U)+\frac{\partial}{\partial y}(\rho r U V)-\rho W^2 &= -r\frac{\partial P}{\partial r}-\frac{\partial}{\partial r}\left(\frac{2}{3}r\rho k\right)+\frac{\partial}{\partial r}\left[2r(\mu+\mu_t)\frac{\partial U}{\partial r}\right]\\ &\quad +\frac{\partial}{\partial y}\left[r(\mu+\mu_t)\left(\frac{\partial U}{\partial y}+\frac{\partial V}{\partial r}\right)\right]\\ &\quad -2(\mu+\mu_t)\frac{U}{r}+\frac{2}{3}\rho k \qquad \text{(A.11)}\end{aligned}$$

Here the $2k/3$ term resulting from the $\overline{w^2}$ Reynolds stress cancels with part of the expanded $\overline{u^2}$ term:

$$\begin{aligned} -\frac{\partial}{\partial r}\left(\frac{2}{3}r\rho k\right)+\frac{2}{3}\rho k &= -r\frac{\partial}{\partial r}\left(\frac{2}{3}\rho k\right)-\frac{2}{3}\rho k\frac{\partial r}{\partial r}+\frac{2}{3}\rho k \\ &= -r\frac{\partial}{\partial r}\left(\frac{2}{3}\rho k\right) \end{aligned} \tag{A.12}$$

The remaining gradient of $2k/3$ is included in the pressure gradient term when the transport equations are coded. The final form of the radial-momentum transport equation is then:

$$\begin{aligned} \frac{\partial}{\partial r}(\rho r U U)+\frac{\partial}{\partial y}(\rho r U V) &= -r\frac{\partial P'}{\partial r}+\frac{\partial}{\partial r}\left(2r\mu_{eff}\frac{\partial U}{\partial r}\right) \\ &+\frac{\partial}{\partial y}\left[r\mu_{eff}\left(\frac{\partial U}{\partial y}+\frac{\partial V}{\partial r}\right)\right]-2\mu_{eff}\frac{U}{r}+\rho W^2 \end{aligned} \tag{A.13}$$

where $\mu_{eff}=\mu+\mu_t$ and $P'=P+2\rho k/3$.

Axial Momentum, V

Following a similar approach outlined above, the axial momentum expression can be written:

$$\begin{aligned} \frac{\partial}{\partial r}(\rho r U V)+\frac{\partial}{\partial y}(\rho r V V) &= -r\frac{\partial P'}{\partial y}+\frac{\partial}{\partial r}\left[r\mu_{eff}\left(\frac{\partial V}{\partial r}+\frac{\partial U}{\partial y}\right)\right] \\ &+\frac{\partial}{\partial y}\left(2r\mu_{eff}\frac{\partial V}{\partial y}\right) \end{aligned} \tag{A.14}$$

Tangential Momentum, W

$$\begin{aligned} \frac{\partial}{\partial r}(\rho r U W)+\frac{\partial}{\partial y}(\rho r V W)+\rho U W &= \frac{\partial}{\partial r}\left(r\mu\frac{\partial W}{\partial r}+r\mu_t\left(\frac{\partial W}{\partial r}-\frac{W}{r}\right)\right)+\frac{\partial}{\partial y}\left(r\mu_{eff}\frac{\partial W}{\partial y}\right) \\ &-\left(\mu\frac{W}{r}+W\frac{\partial \mu}{\partial r}-\mu_t\left(\frac{\partial W}{\partial r}-\frac{W}{r}\right)\right) \end{aligned} \tag{A.15}$$

This can be rearranged as follows:

$$\begin{aligned} \frac{\partial}{\partial r}(\rho r U W)+\frac{\partial}{\partial y}(\rho r V W) &= \frac{\partial}{\partial r}\left(r\mu_{eff}\frac{\partial W}{\partial r}\right)+\frac{\partial}{\partial y}\left(r\mu_{eff}\frac{\partial W}{\partial y}\right) \\ &-\mu_{eff}\frac{W}{r}-W\frac{\partial \mu_{eff}}{\partial r}-\rho U W \end{aligned} \tag{A.16}$$

Kinetic energy, k

$$\begin{aligned}\frac{\partial}{\partial r}(\rho r U k) + \frac{\partial}{\partial y}(\rho r V k) &= \frac{\partial}{\partial r}\left[r\left(\mu + \frac{\mu_t}{\sigma_k}\right)\frac{\partial k}{\partial r}\right] + \frac{\partial}{\partial y}\left[r\left(\mu + \frac{\mu_t}{\sigma_k}\right)\frac{\partial k}{\partial y}\right] \\ &\quad + r P_k - r\rho\varepsilon \end{aligned} \tag{A.17}$$

where the production rate, P_k, is given by:

$$\begin{aligned} P_k &= \mu_t S_{ij}\frac{\partial U_i}{\partial x_j} \\ &= \mu_t \left\{ 2\left(\frac{\partial U}{\partial r}\right)^2 + 2\left(\frac{\partial V}{\partial y}\right)^2 + 2\left(\frac{U}{r}\right)^2 \right. \\ &\quad \left. + \left(\frac{\partial U}{\partial y} + \frac{\partial V}{\partial r}\right)^2 + \left(\frac{\partial W}{\partial r} - \frac{W}{r}\right)^2 + \left(\frac{\partial W}{\partial y}\right)^2 \right\} \end{aligned} \tag{A.18}$$

and the total dissipation rate, ε, comprising of isotropic dissipation rate ($\tilde{\varepsilon}$) and the value of dissipation rate at the wall, is given by:

$$\begin{aligned} \varepsilon &= \tilde{\varepsilon} + 2\nu\left(\frac{\partial\sqrt{k}}{\partial x_j}\right)^2 \\ &= \tilde{\varepsilon} + 2\nu\left[\left(\frac{\partial\sqrt{k}}{\partial r}\right)^2 + \left(\frac{\partial\sqrt{k}}{\partial y}\right)^2\right] \end{aligned} \tag{A.19}$$

Dissipation Rate, $\tilde{\varepsilon}$

$$\begin{aligned}\frac{\partial}{\partial r}(r\rho U \tilde{\varepsilon}) + \frac{\partial}{\partial y}(\rho r V \tilde{\varepsilon}) &= \frac{\partial}{\partial r}\left[r\left(\mu + \frac{\mu_t}{\sigma_\varepsilon}\right)\frac{\partial \tilde{\varepsilon}}{\partial r}\right] + \frac{\partial}{\partial y}\left[r\left(\mu + \frac{\mu_t}{\sigma_\varepsilon}\right)\frac{\partial \tilde{\varepsilon}}{\partial y}\right] \\ &\quad + r\left(c_{\varepsilon 1} f_1 P_k - c_{\varepsilon 2} f_2 \rho\tilde{\varepsilon}\right)\frac{\tilde{\varepsilon}}{k} + r P_{\varepsilon 3} + r\rho Y_c \end{aligned} \tag{A.20}$$

where the gradient production term, $P_{\varepsilon 3}$, is given by:

$$
\begin{aligned}
P_{\varepsilon 3} &= 2\mu\nu_t \left(\frac{\partial^2 U_i}{\partial x_j \partial x_k} \right)^2 \\
&= 2\mu\nu_t \Bigg\{ \left(\frac{\partial^2 U}{\partial r^2} \right)^2 + 3\left[\frac{1}{r^2}\left(r\frac{\partial U}{\partial r} - U \right) \right]^2 + \left(\frac{\partial^2 U}{\partial y^2} \right)^2 \\
&\quad +2\left(\frac{\partial^2 U}{\partial y \partial r} \right)^2 + 2\left(\frac{1}{r}\frac{\partial W}{\partial y} \right)^2 \\
&\quad + \left(\frac{\partial^2 W}{\partial r^2} \right)^2 + 3\left[\frac{1}{r^2}\left(r\frac{\partial W}{\partial r} - W \right) \right]^2 + \left(\frac{\partial^2 W}{\partial y^2} \right)^2 \\
&\quad +2\left(\frac{\partial^2 W}{\partial y \partial r} \right)^2 + 2\left(\frac{1}{r}\frac{\partial U}{\partial y} \right)^2 \\
&\quad + \left(\frac{\partial^2 V}{\partial r^2} \right)^2 + \left(\frac{1}{r}\frac{\partial V}{\partial r} \right)^2 + \left(\frac{\partial^2 V}{\partial y^2} \right)^2 \\
&\quad +2\left(\frac{\partial^2 V}{\partial y \partial r} \right)^2 \Bigg\}
\end{aligned} \tag{A.21}
$$

The whole of the above expression has been used for the work included in this thesis. Morse [124] and Launder & Sharma [13] used a simplified form of the above $P_{\varepsilon 3}$ expression.

A.2 Non-Linear $k - \varepsilon$ Model

In axisymmetric swirling flows the strain-rate and vorticity tensors appearing in the NLEVM are given by:

$$
\begin{aligned}
S_{ij} &= \frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \\
&= \begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{bmatrix} \\
&= \begin{bmatrix} \left(2\frac{\partial U}{\partial r} \right) & \left(\frac{\partial U}{\partial y} + \frac{\partial V}{\partial r} \right) & \left(\frac{\partial W}{\partial r} - \frac{W}{r} \right) \\ \left(\frac{\partial U}{\partial y} + \frac{\partial V}{\partial r} \right) & \left(2\frac{\partial V}{\partial y} \right) & \left(\frac{\partial W}{\partial y} \right) \\ \left(\frac{\partial W}{\partial r} - \frac{W}{r} \right) & \left(\frac{\partial W}{\partial y} \right) & \left(2\frac{U}{r} \right) \end{bmatrix}
\end{aligned} \tag{A.22}
$$

and:

$$
\begin{aligned}
\Omega_{ij} &= \frac{\partial U_i}{\partial x_j} - \frac{\partial U_j}{\partial x_i} \\
&= \begin{bmatrix} \Omega_{11} & \Omega_{12} & \Omega_{13} \\ \Omega_{21} & \Omega_{22} & \Omega_{23} \\ \Omega_{31} & \Omega_{32} & \Omega_{33} \end{bmatrix} \\
&= \begin{bmatrix} 0 & \left(\dfrac{\partial U}{\partial y} - \dfrac{\partial V}{\partial r}\right) & -\left(\dfrac{\partial W}{\partial r} + \dfrac{W}{r}\right) \\[2ex] -\left(\dfrac{\partial U}{\partial y} - \dfrac{\partial V}{\partial r}\right) & 0 & -\left(\dfrac{\partial W}{\partial y}\right) \\[2ex] \left(\dfrac{\partial W}{\partial r} + \dfrac{W}{r}\right) & \left(\dfrac{\partial W}{\partial y}\right) & 0 \end{bmatrix}
\end{aligned}
\tag{A.23}
$$