# Appendix B

# Introduction to Curvilinear Coordinates

## B.1 Definition of a Vector

A vector, $\mathbf{v}$, in three-dimensional space is represented in the most general form as the summation of three components, $v^1$, $v^2$ and $v^3$, aligned with three "base" vectors, as follows:

$$\mathbf{v} = v^1\mathbf{g}_1 + v^2\mathbf{g}_2 + v^3\mathbf{g}_3 = \sum_{i=1}^{3} v^i\mathbf{g}_i \tag{B.1}$$

where bold typeface denotes vector quantities and the base vectors, $\mathbf{g}_i$, can be non-orthogonal and do not have to be unit vectors as long as they are non-coplanar. The subscript $i$ indicates a *covariant* quantity and the superscript $i$ indicates a *contravariant* quantity, hence the above formula describes vector $\mathbf{v}$ as three contravariant components of the covariant base vectors. The Einstein summation convention only applies where one dummy index $i$ is subscript and the other is superscript (summation does not apply over a repeated subscript $i$, so that for instance the metric tensor $g_{ii}$, discussed later, has 3 separate components).

## B.2 Transformation Properties of Covariant and Contravariant Tensors

The subject of covariant and contravariant tensors is often introduced in tensor analysis text books by defining the behaviour of the two under transformation. The gradient of a scalar, $\phi$, is given by the following expression in general non-orthogonal coordinates $(\xi, \eta, \zeta)$:

$$\nabla\phi = \frac{\partial\phi}{\partial\xi}\mathbf{g}^1 + \frac{\partial\phi}{\partial\eta}\mathbf{g}^2 + \frac{\partial\phi}{\partial\zeta}\mathbf{g}^2 = \frac{\partial\phi}{\partial\xi^i}\mathbf{g}^i \tag{B.2}$$

If one defines another coordinate system $\left(\overline{\xi}, \overline{\eta}, \overline{\zeta}\right)$ then components of the gradient can be expressed using the chain-rule:

$$\frac{\partial \phi}{\partial \overline{\xi^i}} = \frac{\partial \xi^j}{\partial \overline{\xi^i}} \frac{\partial \phi}{\partial \xi^j} \tag{B.3}$$

which can be written:

$$\overline{A_i} = a_i^j A_j \tag{B.4}$$

where:

$$\overline{A_i} = \frac{\partial \phi}{\partial \overline{\xi^j}} \qquad a_i^j = \frac{\partial \xi^j}{\partial \overline{\xi^i}} \qquad A_j = \frac{\partial \phi}{\partial \xi^j} \tag{B.5}$$

Tensors that satisfy this transformation are called covariant tensors and have lowered subscripts, as in $A_i$.

To examine the transformation properties of a contravariant tensor, the vector d$\mathbf{r}$ is considered, as follows:

$$\mathrm{d}\mathbf{r} = \mathrm{d}\xi \mathbf{g}_1 + \mathrm{d}\eta \mathbf{g}_2 + \mathrm{d}\zeta \mathbf{g}_3 \tag{B.6}$$

As before, if one defines another coordinate system $\left(\overline{\xi}, \overline{\eta}, \overline{\zeta}\right)$ then components of the vector can be expressed using the chain-rule:

$$\mathrm{d}\overline{\xi^i} = \frac{\partial \overline{\xi^i}}{\partial \xi^j} \mathrm{d}\xi^j \tag{B.7}$$

This can be written:

$$\overline{A^i} = b_j^i A^j \tag{B.8}$$

where:

$$\overline{A^i} \equiv \mathrm{d}\overline{\xi^i} \qquad b_j^i = \frac{\partial \overline{\xi^i}}{\partial \xi^j} \qquad A^j = \mathrm{d}\xi^j \tag{B.9}$$

Tensors that transform according to Equation (B.8) are termed contravariant, and have raised indices.

## B.3 Covariant and Contravariant Base Vectors, $\mathbf{g}_i$ and $\mathbf{g}^i$

One can define a point in space by the position vector, $\mathbf{r}$, using the familiar Cartesian coordinates, as follows:

$$\begin{aligned} \mathbf{r} &= x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}} \\ &= x^1\mathbf{e}_1 + x^2\mathbf{e}_2 + x^3\mathbf{e}_3 \\ &= x^i\mathbf{e}_i \end{aligned} \tag{B.10}$$

and, equally, one can define the unit vector in the $x$-direction, $\hat{\mathbf{i}}$, as follows:

$$\hat{\mathbf{i}} = \frac{\partial \mathbf{r}}{\partial x} \tag{B.11}$$

or, more generally:

$$\mathbf{e}_i = \frac{\partial \mathbf{r}}{\partial x^i} \tag{B.12}$$

The same point in space can be defined using a more general coordinate system:

$$\begin{aligned} \mathbf{r} &= \xi \mathbf{g}_1 + \eta \mathbf{g}_2 + \zeta \mathbf{g}_3 \\ &= \xi^1 \mathbf{g}_1 + \xi^2 \mathbf{g}_2 + \xi^3 \mathbf{g}_3 \\ &= \xi^i \mathbf{g}_i \end{aligned} \tag{B.13}$$

where:

$$\mathbf{g}_i = \frac{\partial \mathbf{r}}{\partial \xi^i} \tag{B.14}$$

Equations (B.10) and (B.13) are equivalent. Using the chain rule, one can therefore express the covariant general base vectors $\mathbf{g}_i$ in terms of the covariant Cartesian base vectors, $\mathbf{e}_i$, as follows:

$$\begin{aligned} \frac{\partial \mathbf{r}}{\partial x^i} &= \frac{\partial \mathbf{r}}{\partial \xi^j} \frac{\partial \xi^j}{\partial x^i} \\ \mathbf{e}_i &= \frac{\partial \xi^j}{\partial x^i} \mathbf{g}_j \end{aligned} \tag{B.15}$$

and likewise:

$$\begin{aligned} \frac{\partial \mathbf{r}}{\partial \xi^i} &= \frac{\partial \mathbf{r}}{\partial x^j} \frac{\partial x^j}{\partial \xi^i} \\ \mathbf{g}_i &= \frac{\partial x^j}{\partial \xi^i} \mathbf{e}_j \end{aligned} \tag{B.16}$$

The covariant and contravariant base vectors are defined such that the scalar product of the covariant and contravariant base vectors is unity, i.e.:

$$\begin{aligned} \mathbf{g}_i \cdot \mathbf{g}^j &= 1 \quad \text{if} \quad i = j \\ &= 0 \quad \text{if} \quad i \neq j \end{aligned}$$

or:

$$\mathbf{g}_i \cdot \mathbf{g}^j = \delta_i^j \tag{B.17}$$

where $\delta_i^j \left(\equiv \delta^{ij} \equiv \delta_{ij}\right)$ is the Kronecker delta.

In Equation (B.13), the vector $\mathbf{r}$ was expressed in terms of the covariant base vector $\mathbf{g}_i$.

In a similar way, vector $\mathbf{r}$ can be written in terms of the contravariant base vector $\mathbf{g}^i$:

$$\mathbf{r} = \xi_i \mathbf{g}^i \tag{B.18}$$

where, following a similar analysis to that given for Equation (B.16):

$$\frac{\partial \mathbf{r}}{\partial \xi_i} = \frac{\partial \mathbf{r}}{\partial x_j} \frac{\partial x_j}{\partial \xi_i} \tag{B.19}$$

and since $\mathbf{g}^i = \partial \mathbf{r}/\partial \xi_i$ and $\mathbf{e}^j = \partial \mathbf{r}/\partial x_j$, the contravariant base vector is given by:

$$\mathbf{g}^i = \frac{\partial x_j}{\partial \xi_i} \mathbf{e}^j \tag{B.20}$$

One can obtain the covariant and contravariant components from the scalar product of the vector, $\mathbf{r}$, and the corresponding base vectors ($\mathbf{g}_i$ or $\mathbf{g}^i$), as follows:

$$\mathbf{r} \cdot \mathbf{g}_i = \xi_j \mathbf{g}^j \cdot \mathbf{g}_i = \xi_j \delta_i^j = \xi_i \tag{B.21}$$

$$\mathbf{r} \cdot \mathbf{g}^i = \xi^j \mathbf{g}_j \cdot \mathbf{g}^i = \xi^j \delta_j^i = \xi^i \tag{B.22}$$

where $\delta_i^j$ has substitution operator properties (i.e. it changes the component $\xi_j$ to $\xi_i$, or from $\xi^j$ to $\xi^i$). Comparing Equations (B.18) and (B.21) one can also see that if the base vector is taken from the right-hand-side to the left-hand-side of Equation (B.18), the superscript $\mathbf{g}^i$ becomes subscript $\mathbf{g}_i$.

There is an alternative method to obtaining the contravariant base vector $\mathbf{g}^i$ as a function of $\mathbf{e}^j$ to that shown above. Returning to Equation (B.15), it was shown that:

$$\mathbf{e}_k = \frac{\partial \xi^j}{\partial x^k} \mathbf{g}_j \tag{B.23}$$

Taking the scalar product of both sides of this equation with $\mathbf{g}^i$:

$$\mathbf{e}_k \cdot \mathbf{g}^i = \frac{\partial \xi^j}{\partial x^k} \mathbf{g}_j \cdot \mathbf{g}^i = \frac{\partial \xi^j}{\partial x^k} \delta_j^i = \frac{\partial \xi^i}{\partial x^k} \tag{B.24}$$

Now, assuming that the contravariant base vector $\mathbf{g}^i$ can be obtained from $\mathbf{e}^j$ using a linear combination of factors $\alpha_j^i$:

$$\mathbf{g}^i = \alpha_1^i \mathbf{e}^1 + \alpha_2^i \mathbf{e}^2 + \alpha_3^i \mathbf{e}^3 = \alpha_j^i \mathbf{e}^j \tag{B.25}$$

and taking the scalar product of both sides of this with $\mathbf{e}_k$:

$$\mathbf{g}^i \cdot \mathbf{e}_k = \alpha_j^i \mathbf{e}^j \cdot \mathbf{e}_k = \alpha_j^i \delta_k^j = \alpha_k^i \tag{B.26}$$

where $\left(\mathbf{g}^i \cdot \mathbf{e}_k = \partial\xi^i/\partial x^k\right)$ from Equation (B.24) and:

$$\alpha^i_j = \frac{\partial \xi^i}{\partial x^j} \tag{B.27}$$

Finally, from Equation (B.25), one obtains:

$$\mathbf{g}^i = \frac{\partial \xi^i}{\partial x^j}\mathbf{e}^j \tag{B.28}$$

which is the same result as Equation (B.20).

The vector product $(\mathbf{g}_2 \times \mathbf{g}_3)$ has magnitude equal to the area of the rectangle[1] with sides $\mathbf{g}_2$ and $\mathbf{g}_3$, with direction $\hat{\mathbf{n}}$ normal to both $\mathbf{g}_2$ and $\mathbf{g}_3$. The scalar product $(\mathbf{g}_1 \cdot \hat{\mathbf{n}})$ is equivalent to a distance in the normal direction, thus the volume of the parallelepiped spanned by vectors $\mathbf{g}_1$, $\mathbf{g}_2$ and $\mathbf{g}_3$ is given by:

$$\Delta Vol = \mathbf{g}_1 \cdot (\mathbf{g}_2 \times \mathbf{g}_3) \tag{B.30}$$

The contravariant base vectors also satisfy:

$$\mathbf{g}^1 = \frac{1}{\Delta Vol}(\mathbf{g}_2 \times \mathbf{g}_3) \quad \mathbf{g}^2 = \frac{1}{\Delta Vol}(\mathbf{g}_3 \times \mathbf{g}_1) \quad \mathbf{g}^3 = \frac{1}{\Delta Vol}(\mathbf{g}_1 \times \mathbf{g}_2) \tag{B.31}$$

and similarly the covariant base vectors satisfy:

$$\mathbf{g}_1 = \frac{1}{\Delta Vol'}\left(\mathbf{g}^2 \times \mathbf{g}^3\right) \quad \mathbf{g}_2 = \frac{1}{\Delta Vol'}\left(\mathbf{g}^3 \times \mathbf{g}^1\right) \quad \mathbf{g}_3 = \frac{1}{\Delta Vol'}\left(\mathbf{g}^1 \times \mathbf{g}^2\right) \tag{B.32}$$

where $\Delta Vol' = \mathbf{g}^1 \cdot \left(\mathbf{g}^2 \times \mathbf{g}^3\right)$ represents the volume of the parallelepiped spanned by the contravariant base vectors $\mathbf{g}^1$, $\mathbf{g}^2$and $\mathbf{g}^3$.

It is useful to note at this point that the covariant and contravariant rectangular Cartesian base vectors are identical, $\mathbf{e}^m \equiv \mathbf{e}_m$. This is partly why covariant and contravariant tensors are not mentioned in most fluid mechanics text books which only deal with Cartesian tensors. The equivalence of covariant and contravariant Cartesian tensors is demonstrated by:

$$\mathbf{g}^1 = \frac{1}{\Delta Vol}(\mathbf{g}_2 \times \mathbf{g}_3) \tag{B.33}$$

which states that the contravariant $\mathbf{g}^1$ vector is perpendicular to the plane defined by the two covariant vectors, $\mathbf{g}_2$ and $\mathbf{g}_3$. In Cartesian coordinates there is no distinction between $\mathbf{g}^1$ and $\mathbf{g}_1$ since the $\hat{\mathbf{k}}$ vector is orthogonal to the plane defined by the $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ vectors (i.e. the $\mathbf{g}_1$

[1]The vector product is defined as:

$$(\mathbf{g}_2 \times \mathbf{g}_3) = (|\mathbf{g}_2|\,|\mathbf{g}_3|\sin\theta)\,\hat{\mathbf{n}} \tag{B.29}$$

where $\hat{\mathbf{n}}$ is the unit normal to vectors $\mathbf{g}_2$ and $\mathbf{g}_3$ and $\theta$ is the angle between the two $\mathbf{g}_2$ and $\mathbf{g}_3$ vectors. Since the area of a triangle with sides $\mathbf{g}_2$ and $\mathbf{g}_3$ is determined from $(1/2 \times base \times height)$ which is equivalent to $(1/2 \times |\mathbf{g}_2| \times |\mathbf{g}_3|\sin\theta)$, the magnitude of the cross product must be equal to the area of the rectangle with sides $\mathbf{g}_2$ and $\mathbf{g}_3$ (i.e. two triangles).

vector is perpendicular to the plane defined by $\mathbf{g}_2$ and $\mathbf{g}_3$).

## B.4 The Jacobian Matrix, $[J]$

It has previously been shown (Equations B.16 and B.28) that the covariant and contravariant base vectors, $\mathbf{g}_i$ and $\mathbf{g}^i$, can be expressed in terms of the Cartesian base vectors, $\mathbf{e}_j$ or $\mathbf{e}^j$, as follows:

$$\mathbf{g}_i = \frac{\partial x^j}{\partial \xi^i}\mathbf{e}_j \tag{B.34}$$

$$\mathbf{g}^i = \frac{\partial \xi^i}{\partial x^j}\mathbf{e}^j \tag{B.35}$$

The Jacobian matrix, $[J]$, is defined as the matrix of coefficients appearing in Equation (B.34):

$$[J] = \frac{\partial x^j}{\partial \xi^i} = \begin{bmatrix} x_\xi & x_\eta & x_\zeta \\ y_\xi & y_\eta & y_\zeta \\ z_\xi & z_\eta & z_\zeta \end{bmatrix} \tag{B.36}$$

where, for example, $x_\xi \equiv \partial x/\partial \xi$ and all components are contravariant, i.e.:

$$\begin{aligned} x &\equiv x^1 \\ y &\equiv x^2 \\ z &\equiv x^3 \\ \xi &\equiv \xi^1 \\ \eta &\equiv \xi^2 \\ \zeta &\equiv \xi^3 \end{aligned}$$

## B.5 Determinant of the Jacobian Matrix, $J$

The Jacobian, $J$, is defined as the determinant of the Jacobian matrix:

$$J = \det[J] = x_\xi\left(y_\eta z_\zeta - y_\zeta z_\eta\right) - x_\eta\left(y_\xi z_\zeta - y_\zeta z_\xi\right) + x_\zeta\left(y_\xi z_\eta - y_\eta z_\xi\right) \tag{B.37}$$

It was noted earlier that the base vectors used to describe vector $\mathbf{r}$ in three-dimensional space should not be coplanar. It was also shown that the volume of the parallelepiped spanned by the base vectors $\mathbf{g}_1$, $\mathbf{g}_2$ and $\mathbf{g}_3$ is given by:

$$\Delta Vol = \mathbf{g}_1 \cdot (\mathbf{g}_2 \times \mathbf{g}_3) \tag{B.38}$$

Using Equation (B.16), the vector product of $\mathbf{g}_2$ and $\mathbf{g}_3$ at a point in space is given by:

$$\begin{aligned}\mathbf{g}_2 \times \mathbf{g}_3 &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ x_\eta & y_\eta & z_\eta \\ x_\zeta & y_\zeta & z_\zeta \end{vmatrix} \\ &= \hat{\mathbf{i}}\left(y_\eta z_\zeta - z_\eta y_\zeta\right) - \hat{\mathbf{j}}\left(x_\eta z_\zeta - z_\eta x_\zeta\right) + \hat{\mathbf{k}}\left(x_\eta y_\zeta - y_\eta x_\zeta\right)\end{aligned} \tag{B.39}$$

and the volume is given by:

$$\begin{aligned}\Delta Vol &= \mathbf{g}_1 \cdot \left(\mathbf{g}_2 \times \mathbf{g}_3\right) \\ &= x_\xi\left(y_\eta z_\zeta - z_\eta y_\zeta\right) - y_\xi\left(x_\eta z_\zeta - z_\eta x_\zeta\right) + z_\xi\left(x_\eta y_\zeta - y_\eta x_\zeta\right)\end{aligned} \tag{B.40}$$

This can be rearranged to give:

$$\Delta Vol = x_\xi\left(y_\eta z_\zeta - y_\zeta z_\eta\right) - x_\eta\left(y_\xi z_\zeta - y_\zeta z_\xi\right) + x_\zeta\left(y_\xi z_\eta - y_\eta z_\xi\right) \tag{B.41}$$

Since Equations (B.37) and (B.41) are identical, the Jacobian, $J$, is equivalent to the cell volume, $\Delta Vol$. Therefore, if the three base vectors are non-coplanar, $J \neq 0$.

## B.6 Inverse of the Jacobian Matrix, $[J]^{-1}$

Taking the scalar product of Equation (B.34) and (B.35):

$$\mathbf{g}_i \cdot \mathbf{g}^k = \frac{\partial x^j}{\partial \xi^i}\mathbf{e}_j \cdot \frac{\partial \xi^k}{\partial x^m}\mathbf{e}^m \tag{B.42}$$

and since $\left(\mathbf{g}_i \cdot \mathbf{g}^k = \delta_i^k\right)$ and $\left(\mathbf{e}_j \cdot \mathbf{e}^m = \delta_j^m\right)$:

$$\begin{aligned}\delta_i^k &= \frac{\partial x^j}{\partial \xi^i}\frac{\partial \xi^k}{\partial x^m}\delta_j^m \\ 1 &= \frac{\partial x^j}{\partial \xi^i}\frac{\partial \xi^i}{\partial x^j}\end{aligned} \tag{B.43}$$

Therefore, if the Jacobian matrix is represented by $\left(\partial x^j/\partial \xi^i\right)$ then the inverse of the Jacobian must be given by $\left(\partial \xi^i/\partial x^j\right)$.

The inverse of the Jacobian matrix is found from:

$$[J]^{-1} = \frac{\partial \xi^i}{\partial x^j} = \begin{bmatrix} \xi_x & \xi_y & \xi_z \\ \eta_x & \eta_y & \eta_z \\ \zeta_x & \zeta_y & \zeta_z \end{bmatrix} = \frac{1}{J}\left[\text{cof }(J)\right]^T \tag{B.44}$$

$$= \frac{1}{J}\begin{bmatrix} (y_\eta z_\zeta - y_\zeta z_\eta) & -(x_\eta z_\zeta - x_\zeta z_\eta) & (x_\eta y_\zeta - x_\zeta y_\eta) \\ -(y_\xi z_\zeta - y_\zeta z_\xi) & (x_\xi z_\zeta - x_\zeta z_\xi) & -(x_\xi y_\zeta - x_\zeta y_\xi) \\ (y_\xi z_\eta - y_\eta z_\xi) & -(x_\xi z_\eta - x_\eta z_\xi) & (x_\xi y_\eta - x_\eta y_\xi) \end{bmatrix} \tag{B.45}$$

where, from the definition of the inverse of a matrix, $[\text{cof }(J)]^T$ is the transpose of the matrix of cofactors of the Jacobian matrix (or adjoint matrix, adj $[J]$).

## B.7 Covariant Metric Tensor, $g_{ij}$

The scalar product of vector $\mathbf{r} = \xi^j \mathbf{g}_j$ with covariant base vector $\mathbf{g}_i$ is as follows:

$$\mathbf{r} \cdot \mathbf{g}_i = \left(\xi^j \mathbf{g}_j\right) \cdot \mathbf{g}_i = \xi^j \left(\mathbf{g}_j \cdot \mathbf{g}_i\right) \tag{B.46}$$

The scalar product of two covariant base vectors $(\mathbf{g}_i \cdot \mathbf{g}_j)$ is termed the covariant "metric tensor", $g_{ij}$. Due to the symmetry of the scalar product, the metric tensor is symmetrical:

$$g_{ij} = \mathbf{g}_i \cdot \mathbf{g}_j = \mathbf{g}_j \cdot \mathbf{g}_i = g_{ji} \tag{B.47}$$

The action of the covariant metric tensor $g_{ij}$ is often referred to as "lowering the index", where scaling a contravariant component $\xi^j$ with the metric tensor $g_{ij}$ effectively lowers the index to give a covariant component $\xi_i$:

$$\xi_i = g_{ij}\xi^j \tag{B.48}$$

The above equation can be derived by considering the scalar product of vector $\mathbf{r}$ and $\mathbf{g}_i$, assuming the vector $\mathbf{r}$ to be given by $\xi_j \mathbf{g}^j$:

$$\mathbf{r} \cdot \mathbf{g}_i = \left(\xi_j \mathbf{g}^j\right) \cdot \mathbf{g}_i = \xi_j \delta_i^j = \xi_i \tag{B.49}$$

which is equivalent to Equation (B.46):

$$\mathbf{r} \cdot \mathbf{g}_i = \xi^j g_{ij} \tag{B.50}$$

Using Equation (B.34), the metric tensor can be written:

$$g_{ij} = \mathbf{g}_i \cdot \mathbf{g}_j = \frac{\partial x^k}{\partial \xi^i}\mathbf{e}_k \cdot \frac{\partial x^m}{\partial \xi^j}\mathbf{e}_m \tag{B.51}$$

and, since $\mathbf{e}_k$ and $\mathbf{e}_m$ are Cartesian base vectors ($\mathbf{e}_k \cdot \mathbf{e}_m = \delta_{km}$):

$$\begin{aligned} g_{ij} &= \frac{\partial x^k}{\partial \xi^i}\frac{\partial x^m}{\partial \xi^j}\delta_{km} \\ &= \sum_{k=1}^{3}\frac{\partial x^k}{\partial \xi^i}\frac{\partial x^k}{\partial \xi^j} \\ &= \frac{\partial x}{\partial \xi^i}\frac{\partial x}{\partial \xi^j} + \frac{\partial y}{\partial \xi^i}\frac{\partial y}{\partial \xi^j} + \frac{\partial z}{\partial \xi^i}\frac{\partial z}{\partial \xi^j} \end{aligned} \tag{B.52}$$

Using this definition of the covariant metric tensor, and Equation (B.28), one can also show that $g_{ij}$ is capable of lowering the index of a vector. The product of the metric $g_{ij}$ and the contravariant base vector $\mathbf{g}^j$ can be expanded as follows:

$$g_{ij}\mathbf{g}^j = \left(\frac{\partial x^k}{\partial \xi^i}\frac{\partial x^k}{\partial \xi^j}\right)\left(\frac{\partial \xi^j}{\partial x^m}\mathbf{e}^m\right) \tag{B.53}$$

Simplifying, using the chain-rule:

$$\begin{aligned} g_{ij}\mathbf{g}^j &= \frac{\partial x^k}{\partial \xi^i}\frac{\partial x^k}{\partial x^m}\mathbf{e}^m \\ &= \frac{\partial x^k}{\partial \xi^i}\delta^k_m\mathbf{e}^m \end{aligned} \tag{B.54}$$

and, since the covariant and contravariant rectangular Cartesian base vectors are identical, $\mathbf{e}^m = \mathbf{e}_m$, then from Equation (B.16):

$$g_{ij}\mathbf{g}^j = \frac{\partial x^k}{\partial \xi^i}\mathbf{e}_k = \mathbf{g}_i \tag{B.55}$$

## B.8 Determinant of the Covariant Metric Tensor Matrix, $g$

Using Equation (B.52), the covariant metric tensor matrix can be written:

$$\begin{aligned} [g_{ij}] &= \begin{bmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{bmatrix} \\ &= \begin{bmatrix} (x_\xi x_\xi + y_\xi y_\xi + z_\xi z_\xi) & (x_\xi x_\eta + y_\xi y_\eta + z_\xi z_\eta) & (x_\xi x_\zeta + y_\xi y_\zeta + z_\xi z_\zeta) \\ (x_\eta x_\xi + y_\eta y_\xi + z_\eta z_\xi) & (x_\eta x_\eta + y_\eta y_\eta + z_\eta z_\eta) & (x_\eta x_\zeta + y_\eta y_\zeta + z_\eta z_\zeta) \\ (x_\zeta x_\xi + y_\zeta y_\xi + z_\zeta z_\xi) & (x_\zeta x_\eta + y_\zeta y_\eta + z_\zeta z_\eta) & (x_\zeta x_\zeta + y_\zeta y_\zeta + z_\zeta z_\zeta) \end{bmatrix} \end{aligned} \tag{B.56}$$

This is equivalent to the product of the Jacobian matrix and the transpose of the Jacobian matrix:

$$\begin{aligned} [g_{ij}] &= [J]^T [J] \\ &= \begin{bmatrix} x_\xi & x_\eta & x_\zeta \\ y_\xi & y_\eta & y_\zeta \\ z_\xi & z_\eta & z_\zeta \end{bmatrix}^T \begin{bmatrix} x_\xi & x_\eta & x_\zeta \\ y_\xi & y_\eta & y_\zeta \\ z_\xi & z_\eta & z_\zeta \end{bmatrix} \\ &= \begin{bmatrix} x_\xi & y_\xi & z_\xi \\ x_\eta & y_\eta & z_\eta \\ x_\zeta & y_\zeta & z_\zeta \end{bmatrix} \begin{bmatrix} x_\xi & x_\eta & x_\zeta \\ y_\xi & y_\eta & y_\zeta \\ z_\xi & z_\eta & z_\zeta \end{bmatrix} \end{aligned} \tag{B.57}$$

Using $g$ to denote the determinant of matrix $[g_{ij}]$ one therefore finds that:

$$g = \det(g_{ij}) = \det\left([J]^T [J]\right) = \det[J] \det[J] = J^2 \tag{B.58}$$

where the determinant of a matrix is identical to the determinant of the transpose of the matrix $\left(\det[J] \equiv \det[J]^T\right)$. The above equation can also be written:

$$J = \sqrt{g} \tag{B.59}$$

## B.9 Contravariant Metric Tensor, $g^{ij}$

Following a similar approach to that adopted in Section B.7, one can take the scalar product of vector $\mathbf{r}$ and contravariant base vector $\mathbf{g}^i$, as follows:

$$\mathbf{r} \cdot \mathbf{g}^i = \left(\xi_j \mathbf{g}^j\right) \cdot \mathbf{g}^i = \xi_j \left(\mathbf{g}^j \cdot \mathbf{g}^i\right) = \xi_j g^{ij} \tag{B.60}$$

where $g^{ij}$ is the contravariant metric tensor. Since the scalar product $\mathbf{r} \cdot \mathbf{g}^i$ can also be written:

$$\mathbf{r} \cdot \mathbf{g}^i = \xi^j \mathbf{g}_j \cdot \mathbf{g}^i = \xi^j \delta^i_j = \xi^i \tag{B.61}$$

the actions of the contravariant metric tensor, $g^{ij}$, is often referred to as “raising the index”:

$$\xi^i = g^{ij} \xi_j \tag{B.62}$$

where $\xi_j$ and $\xi^i$ are covariant and contravariant components, respectively.

One can show that the product of the covariant and contravariant metric tensors, $g_{ik}$ and $g^{jk}$, gives the Kronecker delta, $\delta^j_i$, as follows:

$$g_{ik} g^{jk} = (\mathbf{g}_i \cdot \mathbf{g}_k)\left(\mathbf{g}^j \cdot \mathbf{g}^k\right) \tag{B.63}$$

Using the definition:

$$\mathbf{g}_k = \frac{\partial \mathbf{r}}{\partial \xi^k} = \frac{\partial x^m}{\partial \xi^k}\mathbf{e}_m \tag{B.64}$$

and from Equation (B.28):

$$\mathbf{g}^k = \frac{\partial \xi^k}{\partial x^n}\mathbf{e}^n \tag{B.65}$$

one can write the product as:

$$g_{ik}g^{jk} = \left(\frac{\partial x^m}{\partial \xi^i}\frac{\partial x^m}{\partial \xi^k}\right)\left(\frac{\partial \xi^j}{\partial x^n}\frac{\partial \xi^k}{\partial x^n}\right) \tag{B.66}$$

Rearranging these terms:

$$g_{ik}g^{jk} = \left(\frac{\partial x^m}{\partial \xi^i}\frac{\partial \xi^j}{\partial x^n}\right)\left(\frac{\partial x^m}{\partial \xi^k}\frac{\partial \xi^k}{\partial x^n}\right) \tag{B.67}$$

and applying the chain-rule, one obtains:

$$\begin{aligned} g_{ik}g^{jk} &= \left(\frac{\partial x^m}{\partial \xi^i}\frac{\partial \xi^j}{\partial x^n}\right)\left(\frac{\partial x^m}{\partial x^n}\right) \\ &= \left(\frac{\partial x^m}{\partial \xi^i}\frac{\partial \xi^j}{\partial x^n}\right)\delta^m_n \end{aligned} \tag{B.68}$$

Using the substitution operator properties of $\delta^m_n$ and applying once more the chain-rule:

$$\begin{aligned} g_{ik}g^{jk} &= \left(\frac{\partial x^m}{\partial \xi^i}\frac{\partial \xi^j}{\partial x^m}\right) \\ &= \frac{\partial \xi^j}{\partial \xi^i} \end{aligned} \tag{B.69}$$

From the definition of the contravariant metric base vector, $\mathbf{g}_i \cdot \mathbf{g}^j = \delta^j_i$, one obtains:

$$\mathbf{g}_i \cdot \mathbf{g}^j = \frac{\partial x^k}{\partial \xi^i}\mathbf{e}_k \cdot \frac{\partial \xi^j}{\partial x^m}\mathbf{e}^m = \frac{\partial x^k}{\partial \xi^i}\frac{\partial \xi^j}{\partial x^m}\delta^m_k = \frac{\partial \xi^j}{\partial \xi^i} = \delta^j_i \tag{B.70}$$

and therefore:

$$g_{ik}g^{jk} = \delta^j_i \tag{B.71}$$

The matrix of the contravariant metric tensor, $g^{ij}$, is therefore the inverse of the covariant metric tensor, $g_{ij}$, or in terms of matrix manipulation:

$$g^{ij} = \frac{1}{g}G_{ij} \tag{B.72}$$

where $g$ is the determinant and the and $G_{ij}$ is the adjoint of the $g_{ij}$ matrix, given by:

$$\begin{aligned} G_{ij} &= [\text{cof } (g_{ij})]^T \\ &= \begin{bmatrix} (g_{22}g_{33} - g_{23}g_{32}) & -(g_{12}g_{33} - g_{13}g_{32}) & (g_{12}g_{23} - g_{13}g_{22}) \\ -(g_{21}g_{33} - g_{23}g_{31}) & (g_{11}g_{33} - g_{13}g_{31}) & -(g_{11}g_{23} - g_{13}g_{21}) \\ (g_{21}g_{32} - g_{22}g_{31}) & -(g_{11}g_{32} - g_{12}g_{31}) & (g_{11}g_{22} - g_{12}g_{21}) \end{bmatrix} \end{aligned} \quad \text{(B.73)}$$

Since the metric tensor is symmetric ($g_{ij} = g_{ji}$), the adjoint and the matrix of cofactors of the $g_{ij}$ matrix are equivalent, i.e. $[\text{cof } (g_{ij})]^T = \text{cof } (g_{ij})$.

## B.10 Second Order Tensors, T

Second order tensors are represented in general coordinates as follows:

$$\mathbf{T} = T^{ij}\mathbf{g}_i \otimes \mathbf{g}_j = T_{ij}\mathbf{g}^i \otimes \mathbf{g}^j \quad \text{(B.74)}$$

where $\mathbf{g}_i \otimes \mathbf{g}_j$ and $\mathbf{g}^i \otimes \mathbf{g}^j$ are, respectively, the tensor product (or dyadic) of the covariant and contravariant base vectors, and $T^{ij}$ and $T_{ij}$ are, respectively, the contravariant and covariant tensor components.

## B.11 Christoffel Symbols of the First Kind, $\Gamma_{ijk}$

Since the base vectors are generally not constant except in the case of the Cartesian coordinate system, the derivatives of the base vectors also form vectors which characterize the curvature of the curvilinear coordinate system. The vector $\partial \mathbf{g}_i / \partial \xi^j$ is expressed as a linear combination of the contravariant base vectors as follows:

$$\frac{\partial \mathbf{g}_i}{\partial \xi^j} = \Gamma_{ij1}\mathbf{g}^1 + \Gamma_{ij2}\mathbf{g}^2 + \Gamma_{ij3}\mathbf{g}^3 = \Gamma_{ijk}\mathbf{g}^k \quad \text{(B.75)}$$

where $\Gamma_{ijk}$ are the "Christoffel symbols of the first kind". Equation (B.75) can be rearranged to give:

$$\Gamma_{ijk} = \frac{\partial \mathbf{g}_i}{\partial \xi^j} \cdot \mathbf{g}_k \quad \text{(B.76)}$$

If the base vector is given by:

$$\mathbf{g}_i = \frac{\partial \mathbf{r}}{\partial \xi^i} \quad \text{(B.77)}$$

then:

$$\frac{\partial \mathbf{g}_i}{\partial \xi^j} = \frac{\partial^2 \mathbf{r}}{\partial \xi^i \partial \xi^j} = \frac{\partial^2 \mathbf{r}}{\partial \xi^j \partial \xi^i} \quad \text{(B.78)}$$

and hence the $i$ and $j$ subscripts of the Christoffel symbol of the first kind, $\Gamma_{ijk}$, are interchangeable:

$$\Gamma_{ijk} = \Gamma_{jik} \tag{B.79}$$

Differentiating the covariant metric tensor, $g_{ij}$, using the product rule and Equations (B.47) and (B.76), gives:

$$\frac{\partial g_{ij}}{\partial \xi^k} = \mathbf{g}_i \cdot \frac{\partial \mathbf{g}_j}{\partial \xi^k} + \frac{\partial \mathbf{g}_i}{\partial \xi^k} \cdot \mathbf{g}_j = \Gamma_{jki} + \Gamma_{ikj} \tag{B.80}$$

Since $i$, $j$ and $k$ are free indices, one can also write this as:

$$\frac{\partial g_{jk}}{\partial \xi^i} = \Gamma_{kij} + \Gamma_{jik} \tag{B.81}$$

$$\frac{\partial g_{ik}}{\partial \xi^j} = \Gamma_{kji} + \Gamma_{ijk} \tag{B.82}$$

Adding the two equations above, one obtains:

$$2\Gamma_{ijk} + \Gamma_{ikj} + \Gamma_{jki} = \frac{\partial g_{jk}}{\partial \xi^i} + \frac{\partial g_{ik}}{\partial \xi^j} \tag{B.83}$$

and, rearranging this, using Equation (B.80):

$$\Gamma_{ijk} = \frac{1}{2}\left( \frac{\partial g_{jk}}{\partial \xi^i} + \frac{\partial g_{ik}}{\partial \xi^j} - \frac{\partial g_{ij}}{\partial \xi^k} \right) \tag{B.84}$$

The above equation expresses the Christoffel symbol of the first kind as a function of the derivatives of the metric tensor.

## B.12 Christoffel Symbols of the Second Kind, $\Gamma^k_{ij}$

The vector $\partial \mathbf{g}_i / \partial \xi^j$ can also be expressed as a linear combination of the covariant base vectors as follows:

$$\frac{\partial \mathbf{g}_i}{\partial \xi^j} = \Gamma^1_{ij}\mathbf{g}_1 + \Gamma^2_{ij}\mathbf{g}_2 + \Gamma^3_{ij}\mathbf{g}_3 = \Gamma^k_{ij}\mathbf{g}_k \tag{B.85}$$

where the coefficients, $\Gamma^k_{ij}$, are the "Christoffel symbols of the second kind"[2]. Equation (B.85) can be rearranged in terms of the Christoffel symbol:

$$\Gamma^k_{ij} = \frac{\partial \mathbf{g}_i}{\partial \xi^j} \cdot \mathbf{g}^k \tag{B.86}$$

[2] In some texts, the Christoffel symbol of the second kind is written:

$$\Gamma^k_{ij} \equiv \left\{ \begin{matrix} & k & \\ i & & j \end{matrix} \right\}$$

Following a similar method to that used above to derive Equation (B.79), it can be shown that the subscripts of the Christoffel symbol of the second kind are interchangeable, i.e.:

$$\Gamma^k_{ij} = \Gamma^k_{ji} \tag{B.87}$$

Using the product rule to differentiating Equation (B.17), one obtains:

$$\frac{\partial \mathbf{g}_i}{\partial \xi^k} \cdot \mathbf{g}^j + \mathbf{g}_i \cdot \frac{\partial \mathbf{g}^j}{\partial \xi^k} = 0 \tag{B.88}$$

and hence from the definition of the Christoffel symbol of the second kind, Equation (B.86):

$$\Gamma^j_{ik} = -\frac{\partial \mathbf{g}^j}{\partial \xi^k} \cdot \mathbf{g}_i \tag{B.89}$$

Previously it was described how the covariant metric tensor, $g_{ij}$, can lower the index of a tensor. The raised index of the Christoffel symbol of the second kind can also be lowered by the covariant metric tensor, as follows:

$$\Gamma^k_{ij} g_{km} = \frac{\partial \mathbf{g}_i}{\partial \xi^j} \cdot \mathbf{g}^k g_{km} \tag{B.90}$$

From Equation (B.55):

$$\Gamma^k_{ij} g_{km} = \frac{\partial \mathbf{g}_i}{\partial \xi^j} \cdot \mathbf{g}_m \tag{B.91}$$

and from the definition of the Christoffel symbol of the first kind (Equation B.76):

$$\Gamma^k_{ij} g_{km} = \Gamma_{ijm} \tag{B.92}$$

Christoffel symbols of the second kind can therefore be calculated from the metric tensors, using Equation (B.84), as follows:

$$\Gamma^k_{ij} = \frac{1}{2} g^{kl} \left( \frac{\partial g_{jl}}{\partial \xi^i} + \frac{\partial g_{il}}{\partial \xi^j} - \frac{\partial g_{ij}}{\partial \xi^l} \right) \tag{B.93}$$

If this expression is contracted, by setting $k = i$, and has its dummy indices $i$ and $l$ switched around in the last term, one obtains:

$$\begin{aligned} \Gamma^i_{ij} &= \frac{1}{2} g^{il} \frac{\partial g_{jl}}{\partial \xi^i} + \frac{1}{2} g^{il} \frac{\partial g_{il}}{\partial \xi^j} - \frac{1}{2} g^{li} \frac{\partial g_{lj}}{\partial \xi^i} \\ &= \frac{1}{2} g^{il} \frac{\partial g_{il}}{\partial \xi^j} \end{aligned} \tag{B.94}$$

where the first and last terms cancel since the metric tensor is symmetric ($g^{il} = g^{li}$ and $g_{jl} = g_{lj}$).

It is also possible to express Equation (B.94) in terms of the Jacobian $J$ of the metric tensor. The determinant of the metric tensor matrix $[g_{ij}]$ is given by:

$$\det[g_{ij}] = g = g_{11}G_{11} + g_{12}G_{12} + g_{13}G_{13} = g_{11}G_{11} + g_{21}G_{21} + g_{31}G_{31} \tag{B.95}$$

where $G_{ij}$ is the cofactor matrix. Assuming that the determinant $g$ is a function of the nine components of $g_{ij}$, one can obtain the following partial derivatives:

$$\frac{\partial g}{\partial g_{ij}} = G_{ij} \tag{B.96}$$

Using the definition of the inverse of a matrix, Equation (B.72), the above expression can be written:

$$\frac{\partial g}{\partial g_{ij}} = g g^{ij} \tag{B.97}$$

where $g$ and $g^{ij}$ are, respectively, the determinant and the inverse of the matrix of $g_{ij}$. Since the metric tensor is symmetric, the transpose of the metric tensor $\left(g^{ij}\right)^T = g^{ij}$. The term $gg^{ij}$ therefore represents the matrix of cofactors $\left(gg^{ij} = G_{ij}\right)$. The derivative of the determinant, $g$, with respect to the curvilinear coordinates can be written, using the chain rule:

$$\frac{\partial g}{\partial \xi^j} = \frac{\partial g}{\partial g_{il}} \frac{\partial g_{il}}{\partial \xi^j} \tag{B.98}$$

and substituting Equation (B.97):

$$\frac{\partial g}{\partial \xi^j} = g g^{il} \frac{\partial g_{il}}{\partial \xi^j} \tag{B.99}$$

Equation (B.94) can therefore be written:

$$\Gamma_{ij}^{i} = \frac{1}{2} g^{il} \frac{\partial g_{il}}{\partial \xi^j} = \frac{1}{2} \frac{1}{g} \frac{\partial g}{\partial \xi^j} \tag{B.100}$$

By treating the derivative in Equation (B.100) as a "function of a function" and using the definition of the Jacobian $\left(J^2 = g\right)$, the Christoffel can be expressed as:

$$\Gamma_{ij}^{i} = \frac{1}{J} \frac{\partial J}{\partial \xi^j} = \frac{\partial}{\partial \xi^j} (\log J) \tag{B.101}$$

In Cartesian coordinates the base vectors do not vary with position and therefore all components of the Christoffel symbols are zero. It follows then that the Christoffel symbols do not constitute third-order tensors since, if the Christoffel symbols transformed like tensors, their components would remain zero with respect to any coordinate system, which is not the case.

## B.13 Gradient of a Scalar, $\nabla\phi$

The gradient of scalar, $\phi$, is written in curvilinear coordinates:

$$\nabla\phi = \mathbf{g}^j \frac{\partial\phi}{\partial\xi^j} \tag{B.102}$$

The contravariant base vector, $\mathbf{g}^j$, can also be written $g^{jk}\mathbf{g}_k$, using the "raising of the index" property of the contravariant metric, $g^{jk}$. Therefore, the gradient of the scalar can also be written:

$$\nabla\phi = \frac{\partial\phi}{\partial\xi^j} g^{jk}\mathbf{g}_k \tag{B.103}$$

## B.14 Covariant Derivatives of Vectors and Tensors

The covariant derivative of a vector $\left(\mathbf{v} = v^i\mathbf{g}_i\right)$ can be written, using the product rule:

$$\begin{aligned} \frac{\partial\mathbf{v}}{\partial\xi^j} &= \frac{\partial\left(v^i\mathbf{g}_i\right)}{\partial\xi^j} \\ &= \frac{\partial v^i}{\partial\xi^j}\mathbf{g}_i + v^i\frac{\partial\mathbf{g}_i}{\partial\xi^j} \end{aligned} \tag{B.104}$$

and from the definition of the Christoffel symbol (Equation B.86):

$$\begin{aligned} \frac{\partial\mathbf{v}}{\partial\xi^j} &= \frac{\partial v^i}{\partial\xi^j}\mathbf{g}_i + v^i\frac{\partial\mathbf{g}_i}{\partial\xi^j} \\ &= \frac{\partial v^i}{\partial\xi^j}\mathbf{g}_i + v^i\Gamma^k_{ij}\mathbf{g}_k \end{aligned} \tag{B.105}$$

The above derivative is often denoted:

$$\frac{\partial\mathbf{v}}{\partial\xi^j} = \nabla_j v^i\mathbf{g}_i \tag{B.106}$$

or simply:

$$\frac{\partial\mathbf{v}}{\partial\xi^j} = v^i_{,j}\mathbf{g}_i \tag{B.107}$$

where:

$$\nabla_j v^i = v^i_{,j} = \left(\frac{\partial v^i}{\partial\xi^j} + v^k\Gamma^i_{kj}\right) \tag{B.108}$$

The covariant derivative of a second-order tensor $\left(\mathbf{T} = T^{ij}\mathbf{g}_i \otimes \mathbf{g}_j\right)$ is similarly given by:

$$\begin{aligned}\frac{\partial \mathbf{T}}{\partial \xi^k} &= \frac{\partial T^{ij}}{\partial \xi^k}\mathbf{g}_i \otimes \mathbf{g}_j + T^{ij}\frac{\partial \mathbf{g}_i}{\partial \xi^k} \otimes \mathbf{g}_j + T^{ij}\mathbf{g}_i \otimes \frac{\partial \mathbf{g}_j}{\partial \xi^k} \\ &= \frac{\partial T^{ij}}{\partial \xi^k}\mathbf{g}_i \otimes \mathbf{g}_j + T^{ij}\Gamma^m_{ik}\mathbf{g}_m \otimes \mathbf{g}_j + T^{ij}\mathbf{g}_i \otimes \Gamma^m_{jk}\mathbf{g}_m \\ &= \left(\frac{\partial T^{ij}}{\partial \xi^k} + T^{mj}\Gamma^i_{mk} + T^{im}\Gamma^j_{mk}\right)\mathbf{g}_i \otimes \mathbf{g}_j \end{aligned} \tag{B.109}$$

This is often denoted:

$$\frac{\partial \mathbf{T}}{\partial \xi^k} = \nabla_k T^{ij}\mathbf{g}_i \otimes \mathbf{g}_j \tag{B.110}$$

or:

$$\frac{\partial \mathbf{T}}{\partial \xi^k} = T^{ij}_{,k}\mathbf{g}_i \otimes \mathbf{g}_j \tag{B.111}$$

An alternative approach to obtaining the covariant derivative of a second-order tensor is to compare the covariant derivative a scalar which is created from a tensor invariant with the set of its partial derivatives. For example, to find an expression for the covariant derivative of the covariant tensor $T_{ij}$, if one considers first the covariant derivative of scalar $\left(T_{ij}u^i v^j\right)$:

$$\left(T_{ij}u^i v^j\right)_{,k} = T_{ij,k}u^i v^i + T_{ij}\left(\frac{\partial u^i}{\partial \xi^k} + \Gamma^i_{km}u^m\right)v^j + T_{ij}u^i\left(\frac{\partial v^j}{\partial \xi^k} + \Gamma^j_{km}v^m\right) \tag{B.112}$$

The partial derivatives of $\left(T_{ij}u^i v^j\right)$ are given by:

$$\frac{\partial}{\partial \xi^k}\left(T_{ij}u^i v^j\right) = \frac{\partial T_{ij}}{\partial \xi^k}u^i v^j + T_{ij}\frac{\partial u^i}{\partial \xi^k}v^j + T_{ij}u^i\frac{\partial v^j}{\partial \xi^k} \tag{B.113}$$

For a scalar, the above covariant derivative and the set of partial derivatives are identical. Therefore, canceling terms and rearranging:

$$\left(T_{ij,k} + \Gamma^m_{ik}T_{mj} + \Gamma^m_{jk}T_{im} - \frac{\partial T_{ij}}{\partial \xi^k}\right)u^i v^j = 0 \tag{B.114}$$

Finally, canceling the arbitrary terms $u^i$ and $v^j$ and rearranging once more, one obtains:

$$T_{ij,k} = \frac{\partial T_{ij}}{\partial \xi^k} - \Gamma^m_{ik}T_{mj} - \Gamma^m_{jk}T_{im} \tag{B.115}$$

## B.15 Covariant Derivative of the Metric Tensor

The covariant derivative of the Kronecker delta ($\delta_{ij,k}$) is zero. Since $\delta_{ij}$ is equivalent to the metric tensor, $g_{ij}$, in Cartesian coordinates, the covariant derivative of the metric tensor must also be zero:

$$g_{ij,k} = 0 \tag{B.116}$$

This result can be confirmed using the above expression for the covariant derivative of a second-order tensor (Equation B.115):

$$g_{ij,k} = \frac{\partial g_{ij}}{\partial \xi^k} - \Gamma^m_{ik} g_{mj} - \Gamma^m_{jk} g_{im} \tag{B.117}$$

which can also be written using Equation (B.92):

$$g_{ij,k} = \frac{\partial g_{ij}}{\partial \xi^k} - \Gamma_{ikj} - \Gamma_{jki} \tag{B.118}$$

Earlier it was shown that the derivative of the covariant metric tensor $g_{ij}$ with respect to $\xi^k$ is as follows:

$$\begin{aligned} \frac{\partial g_{ij}}{\partial \xi^k} &= \mathbf{g}_i \cdot \frac{\partial \mathbf{g}_j}{\partial \xi^k} + \frac{\partial \mathbf{g}_i}{\partial \xi^k} \cdot \mathbf{g}_j \\ &= \Gamma_{jki} + \Gamma_{ikj} \end{aligned} \tag{B.119}$$

Substituting this expression into Equation (B.118), one obtains:

$$g_{ij,k} = \Gamma_{jki} + \Gamma_{ikj} - \Gamma_{ikj} - \Gamma_{jki} = 0 \tag{B.120}$$

For the special case where $i = j$ one can also write:

$$g_{ii,k} = \frac{\partial g_{ii}}{\partial \xi^k} - 2\Gamma^m_{ik} g_{im} = 0 \tag{B.121}$$

where, following the summation convention, there is no summation on $i$.

One can also show that $g^{ij}_{,k} = 0$, following the same method as described above. The covariant derivative of $g^{ij}$ is expanded:

$$g^{ij}_{,k} = \frac{\partial g^{ij}}{\partial \xi^k} + g^{mj}\Gamma^i_{mk} + g^{im}\Gamma^j_{mk} \tag{B.122}$$

and the set of partial derivatives of $g^{ij}$ :

$$\frac{\partial g^{ij}}{\partial \xi^k} = \mathbf{g}^i \cdot \frac{\partial \mathbf{g}^j}{\partial \xi^k} + \mathbf{g}^j \cdot \frac{\partial \mathbf{g}^i}{\partial \xi^k} \tag{B.123}$$

Using Equation (B.89), the partial derivatives can be expressed in terms of Christoffel symbols of the second kind, as follows:

$$\begin{aligned} \frac{\partial g^{ij}}{\partial \xi^k} &= -\mathbf{g}^i \cdot \mathbf{g}^m \Gamma^j_{mk} - \mathbf{g}^j \cdot \mathbf{g}^m \Gamma^i_{mk} \\ &= -g^{im}\Gamma^j_{mk} - g^{jm}\Gamma^i_{mk} \end{aligned} \tag{B.124}$$

Finally, substituting Equation (B.124) into (B.122):

$$g^{ij}_{,k} = g^{mj}\Gamma^i_{mk} + g^{im}\Gamma^j_{mk} - g^{im}\Gamma^j_{mk} - g^{jm}\Gamma^i_{mk} = 0 \tag{B.125}$$

## B.16 Gradient of a Vector, $\nabla\mathbf{v}$

The gradient of a vector, $\mathbf{v}$, is written:

$$\operatorname{grad}\mathbf{v} = \nabla\mathbf{v} = \mathbf{g}^j\frac{\partial\mathbf{v}}{\partial\xi^j} \tag{B.126}$$

Expanding the derivative term using Equation (B.108):

$$\nabla\mathbf{v} = \left(\frac{\partial v^i}{\partial\xi^j} + v^k\Gamma^i_{kj}\right)\mathbf{g}_i\otimes\mathbf{g}^j \tag{B.127}$$

and using the contravariant metric tensor, $g^{jm}$, to write the gradient in terms of two covariant base vectors, one obtains:

$$\nabla\mathbf{v} \;=\; \left(\frac{\partial v^i}{\partial\xi^j} + v^k\Gamma^i_{kj}\right)g^{jm}\mathbf{g}_i\otimes\mathbf{g}_m \tag{B.128}$$

## B.17 Divergence of a Vector, $\nabla\cdot\mathbf{v}$

The divergence of a vector $\mathbf{v}$ is the derivative of the components of vector $\mathbf{v}$ in each of the respective component directions. This is calculated from the scalar (or inner) product of the *grad* operator ($\nabla$) and the vector $\mathbf{v}$, which can be expressed as:

$$\operatorname{div}\mathbf{v} = \nabla\cdot\mathbf{v} = \mathbf{g}^j\cdot\frac{\partial\mathbf{v}}{\partial\xi^j} = \mathbf{g}^j\cdot\mathbf{v}_{,j} \tag{B.129}$$

Since $\nabla\cdot\mathbf{v}$ consists of a scalar product between two vectors, the resulting expression is a scalar quantity. In Cartesian coordinates, where $\mathbf{v} = u^i\mathbf{e}_i$, the divergence is simply:

$$\begin{aligned}
\nabla\cdot\mathbf{v} &= \mathbf{e}^j\cdot\frac{\partial\left(u^i\mathbf{e}_i\right)}{\partial x^j} \\
&= \mathbf{e}^j\cdot\left(\mathbf{e}_i\frac{\partial u^i}{\partial x^j} + u^i\frac{\partial\mathbf{e}_i}{\partial x^j}\right) \\
&= \delta^j_i\frac{\partial u^i}{\partial x^j} \\
&= \frac{\partial u^i}{\partial x^i} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}
\end{aligned} \tag{B.130}$$

where the derivative $\partial\mathbf{e}_i/\partial x^j$ is zero, since the Cartesian base vectors do not vary with position.

In general non-orthogonal curvilinear coordinates, where $\mathbf{v} = v^i \mathbf{g}_i$, the divergence is given by (see also Section B.14):

$$\begin{aligned}
\nabla \cdot \mathbf{v} &= \mathbf{g}^j \cdot \frac{\partial \left( v^i \mathbf{g}_i \right)}{\partial \xi^j} \\
&= \mathbf{g}^j \cdot \left( \frac{\partial v^i}{\partial \xi^j} \mathbf{g}_i + v^i \Gamma^k_{ij} \mathbf{g}_k \right) \\
&= \delta^j_i \frac{\partial v^i}{\partial \xi^j} + \delta^j_k v^i \Gamma^k_{ij} \\
&= \frac{\partial v^j}{\partial \xi^j} + v^k \Gamma^j_{kj} \qquad \text{(B.131)}
\end{aligned}$$

Using Equation (B.101) this becomes:

$$\nabla \cdot \mathbf{v} = \frac{\partial v^j}{\partial \xi^j} + \frac{v^j}{J} \frac{\partial J}{\partial \xi^j} \qquad \text{(B.132)}$$

and applying the product rule:

$$\nabla \cdot \mathbf{v} = \frac{\Delta v^j}{\Delta \xi^j} = \frac{1}{J} \frac{\partial}{\partial \xi^j} \left( J v^j \right) \qquad \text{(B.133)}$$

## B.18 Divergence of a Tensor, $\nabla \cdot \mathbf{T}$

The divergence of the second-order tensor $\mathbf{T}$, where $\mathbf{T} = T^{ij} \mathbf{g}_i \otimes \mathbf{g}_j$, can be written as follows:

$$\operatorname{div} \mathbf{T} = \nabla \cdot \mathbf{T} = \mathbf{g}^k \cdot \frac{\partial \mathbf{T}}{\partial \xi^k} = \mathbf{g}^k \cdot \mathbf{T}_{,k} \qquad \text{(B.134)}$$

From Equation (B.109), this can be expanded:

$$\begin{aligned}
\nabla \cdot \mathbf{T} &= \mathbf{g}^k \cdot \left( \frac{\partial T^{ij}}{\partial \xi^k} + T^{mj} \Gamma^i_{mk} + T^{im} \Gamma^j_{mk} \right) \mathbf{g}_i \otimes \mathbf{g}_j \qquad \text{(B.135)} \\
&= \delta^k_j \left( \frac{\partial T^{ij}}{\partial \xi^k} + T^{mj} \Gamma^i_{mk} + T^{im} \Gamma^j_{mk} \right) \mathbf{g}_i \qquad \text{(B.136)} \\
&= \left( \frac{\partial T^{ij}}{\partial \xi^j} + T^{mj} \Gamma^i_{mj} + T^{im} \Gamma^j_{mj} \right) \mathbf{g}_i \qquad \text{(B.137)}
\end{aligned}$$

and simplified, using Equation (B.101):

$$\begin{aligned}
\nabla \cdot \mathbf{T} &= \left( \frac{\partial T^{ij}}{\partial \xi^j} + T^{mj}\Gamma^i_{mj} + \frac{T^{ij}}{J}\frac{\partial J}{\partial \xi^j} \right) \mathbf{g}_i && \text{(B.138)} \\
&= \left[ \frac{1}{J}\frac{\partial}{\partial \xi^j}\left(JT^{ij}\right) + T^{mj}\Gamma^i_{mj} \right] \mathbf{g}_i && \text{(B.139)} \\
&= \left( \frac{\Delta T^{ij}}{\Delta \xi^j} + T^{mj}\Gamma^i_{mj} \right) \mathbf{g}_i && \text{(B.140)} \\
&= \nabla_j T^{ij} \mathbf{g}_i && \text{(B.141)}
\end{aligned}$$

Note that the divergence of a second-order tensor results in a vector quantity. In going from Equation (B.135) to Equation (B.136) the base vectors $\mathbf{g}^k$ and $\mathbf{g}_i$ were combined into the Kronecker delta $\left(\mathbf{g}^k \cdot \mathbf{g}_i = \delta^k_i\right)$. One could equally have combined $\mathbf{g}^k$ and $\mathbf{g}_j$ which would give exactly the same result with different dummy indices.

## B.19 Summation Convention

It was shown earlier that the divergence of a vector, $\mathbf{v}$, is given by:

$$\nabla \cdot \mathbf{v} = v^j_{,j} = \frac{\partial v^j}{\partial \xi^j} + v^k \Gamma^j_{kj} \tag{B.142}$$

If the vector $\mathbf{v}$ is assumed to be obtained from the gradient of a scalar field ($\mathbf{v} = \operatorname{grad} \phi$), then the divergence of vector $\mathbf{v}$ is given by:

$$\operatorname{div}\,(\operatorname{grad} \phi) = \nabla \cdot (\nabla \phi) = \nabla^2 \phi \tag{B.143}$$

where $\nabla^2$ is the Laplacian operator. The covariant derivative of the scalar field is given by:

$$\frac{\partial \phi}{\partial \xi^j} = \phi_{,j} \tag{B.144}$$

however, in general coordinate systems, one cannot write:

$$\nabla \cdot (\nabla \phi) = \left(\phi_{,j}\right)_{,j} \tag{B.145}$$

since summation can only be applied with indices on different levels. Instead, one needs to first calculate the contravariant component of $\operatorname{grad} \phi$, by multiplying with the metric tensor $g^{ij}$, before taking the divergence, as follows:

$$\begin{aligned}
\nabla \cdot (\nabla \phi) &= \left(g^{ij}\phi_{,j}\right)_{,i} \\
&= \frac{\partial}{\partial \xi^i}\left(g^{ij}\frac{\partial \phi}{\partial \xi^j}\right) + \Gamma^i_{ki} g^{kj} \frac{\partial \phi}{\partial \xi^j} && \text{(B.146)}
\end{aligned}$$

This now satisfies the summation convention as indices $i$, $j$ and $k$ are repeated in upper and lower positions.

## B.20 Physical Components

In the preceding analysis, tensor calculus has been used to express the divergence of first- and second-order tensors in terms of the covariant and contravariant base vectors, $\mathbf{g}_i$ and $\mathbf{g}^i$. In some coordinate systems the dimensions of the components of the base vectors are different to those of their parents (e.g. in cylindrical-polar coordinates ($r$, $y$, $\phi$) the velocity component in the direction of the angular coordinate $\phi$ does not have dimensions of length/time). To overcome this shortcoming, unit base vectors are used, which are obtained by dividing each base vector by its magnitude[3], i.e.:

$$\mathbf{g}_{(i)} = \frac{1}{\sqrt{g_{ii}}}\mathbf{g}_i \tag{B.148}$$

which can be rearranged to give:

$$\mathbf{g}_i = \sqrt{g_{ii}}\mathbf{g}_{(i)} \tag{B.149}$$

where there is no summation over the repeated $i$ index in the above equations. A vector $\mathbf{v}$ can therefore be expressed as follows:

$$\mathbf{v} = v^i\mathbf{g}_i = v^i\sqrt{g_{ii}}\mathbf{g}_{(i)} \tag{B.150}$$

Since the vector $\mathbf{g}_{(i)}$ is a unit vector, the components $\left(v^i\sqrt{g_{ii}}\right)$ must have the same dimensions as the parent vector $\mathbf{v}$. These components are therefore called the "physical" components, denoted $v^{(i)}$. First- and second-order tensors can be expressed in terms of the physical components as follows:

$$\mathbf{v} = v^{(i)}\mathbf{g}_{(i)} = v_{(i)}\mathbf{g}^{(i)} \tag{B.151}$$

$$\mathbf{T} = T^{(ij)}\mathbf{g}_{(i)} \otimes \mathbf{g}_{(j)} \tag{B.152}$$

where:

$$v^{(i)} = \sqrt{g_{ii}}v^i \quad \text{(no summation)} \tag{B.153}$$

$$T^{(ij)} = \sqrt{g_{ii}}\sqrt{g_{jj}}T^{ij} \quad \text{(no summation)} \tag{B.154}$$

## B.21 Key Formulae

A summary of equations derived in the preceding sections is given below:

[3] The magnitude of a vector can be found from the scalar product ($\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}|\cos\theta$), so for the covariant base vector the magnitude is given by:

$$\sqrt{g_{ii}} = \sqrt{\mathbf{g}_i \cdot \mathbf{g}_i} = \sqrt{|\mathbf{g}_i|\,|\mathbf{g}_i|\cos 0} = |\mathbf{g}_i| \tag{B.147}$$

- Covariant Metric Tensor, $g_{ij}$:

$$g_{ij} = \sum_{k=1}^{3} \frac{\partial x^k}{\partial \xi^i} \frac{\partial x^k}{\partial \xi^j} = \frac{\partial x}{\partial \xi^i} \frac{\partial x}{\partial \xi^j} + \frac{\partial y}{\partial \xi^i} \frac{\partial y}{\partial \xi^j} + \frac{\partial z}{\partial \xi^i} \frac{\partial z}{\partial \xi^j} \tag{B.155}$$

- Jacobian Matrix, $[J]$:

$$[J] = \frac{\partial x^j}{\partial \xi^i} = \begin{bmatrix} x_\xi & x_\eta & x_\zeta \\ y_\xi & y_\eta & y_\zeta \\ z_\xi & z_\eta & z_\zeta \end{bmatrix} \tag{B.156}$$

- Jacobian, $J$:

$$J = \det [J] = x_\xi \left(y_\eta z_\zeta - y_\zeta z_\eta\right) - x_\eta \left(y_\xi z_\zeta - y_\zeta z_\xi\right) + x_\zeta \left(y_\xi z_\eta - y_\eta z_\xi\right) \tag{B.157}$$

- Inverse Jacobian Matrix, $[J]^{-1}$:

$$[J]^{-1} = \frac{\partial \xi^i}{\partial x^j} = \begin{bmatrix} \xi_x & \xi_y & \xi_z \\ \eta_x & \eta_y & \eta_z \\ \zeta_x & \zeta_y & \zeta_z \end{bmatrix} = \frac{1}{J} \left[\operatorname{cof}(J)\right]^T \tag{B.158}$$

$$= \frac{1}{J} \begin{bmatrix} \left(y_\eta z_\zeta - y_\zeta z_\eta\right) & -\left(x_\eta z_\zeta - x_\zeta z_\eta\right) & \left(x_\eta y_\zeta - x_\zeta y_\eta\right) \\ -\left(y_\xi z_\zeta - y_\zeta z_\xi\right) & \left(x_\xi z_\zeta - x_\zeta z_\xi\right) & -\left(x_\xi y_\zeta - x_\zeta y_\xi\right) \\ \left(y_\xi z_\eta - y_\eta z_\xi\right) & -\left(x_\xi z_\eta - x_\eta z_\xi\right) & \left(x_\xi y_\eta - x_\eta y_\xi\right) \end{bmatrix} \tag{B.159}$$

- Contravariant Metric Tensor, $g^{ij}$:

$$g^{ij} = \frac{1}{g} G_{ij} \tag{B.160}$$

- Determinant of Covariant Metric Tensor Matrix, $g$:

$$g = J^2 \tag{B.161}$$

- Adjoint of the Covariant Metric Tensor Matrix, $G_{ij}$:

$$\begin{aligned} G_{ij} &= \left[\operatorname{cof}(g_{ij})\right]^T \\ &= \begin{bmatrix} \left(g_{22}g_{33} - g_{23}g_{32}\right) & -\left(g_{12}g_{33} - g_{13}g_{32}\right) & \left(g_{12}g_{23} - g_{13}g_{22}\right) \\ -\left(g_{21}g_{33} - g_{23}g_{31}\right) & \left(g_{11}g_{33} - g_{13}g_{31}\right) & -\left(g_{11}g_{23} - g_{13}g_{21}\right) \\ \left(g_{21}g_{32} - g_{22}g_{31}\right) & -\left(g_{11}g_{32} - g_{12}g_{31}\right) & \left(g_{11}g_{22} - g_{12}g_{21}\right) \end{bmatrix} \end{aligned} \tag{B.162}$$

- Christoffel Symbol of the Second Kind, $\Gamma^k_{ij}$:

$$\Gamma^k_{ij} = \frac{1}{2} g^{kl} \left( \frac{\partial g_{jl}}{\partial \xi^i} + \frac{\partial g_{il}}{\partial \xi^j} - \frac{\partial g_{ij}}{\partial \xi^l} \right) \tag{B.163}$$

- Covariant Derivative, $\nabla_j v^i$:

$$\nabla_j v^i = \left( \frac{\partial v^i}{\partial \xi^j} + v^k \Gamma^i_{kj} \right) \tag{B.164}$$

- Gradient of a Scalar, $\nabla \phi$:

$$\nabla \phi = \frac{\partial \phi}{\partial \xi^j} g^{jk} \mathbf{g}_k \tag{B.165}$$

- Divergence of a Vector, $\nabla \cdot \mathbf{v}$:

$$\nabla \cdot \mathbf{v} = \frac{\Delta v^j}{\Delta \xi^j} = \frac{1}{J} \frac{\partial}{\partial \xi^j} \left( J v^j \right) \tag{B.166}$$

- Divergence of a Tensor, $\nabla \cdot \mathbf{T}$:

$$\begin{aligned} \nabla \cdot \mathbf{T} &= \nabla_j T^{ij} \mathbf{g}_i \\ &= \left( \frac{\Delta T^{ij}}{\Delta \xi^j} + T^{mj} \Gamma^i_{mj} \right) \mathbf{g}_i \\ &= \left[ \frac{1}{J} \frac{\partial}{\partial \xi^j} \left( J T^{ij} \right) + T^{mj} \Gamma^i_{mj} \right] \mathbf{g}_i \end{aligned} \tag{B.167}$$

- Physical Vector Component, $v^{(i)}$:

$$v^{(i)} = \sqrt{g_{ii}} v^i \quad \text{(no summation)} \tag{B.168}$$

- Physical Tensor Component, $T^{(ij)}$:

$$T^{(ij)} = \sqrt{g_{ii}} \sqrt{g_{jj}} T^{ij} \quad \text{(no summation)} \tag{B.169}$$