Appendix D

Subgrid Wall Function Transport Equations

In the subgrid wall function a number of assumptions are made to obtain a set of simplified transport equations:

- Only the velocity components parallel to the wall are solved.
- Diffusion parallel to the wall is assumed to be negligible in comparison with diffusion normal to the wall.
- Convection is modelled in non-conservative form.
- The velocity component normal to the wall is obtained from continuity across the subgrid control volumes, with an additional scaling factor to ensure consistency in the boundary conditions.

For the present purposes of deriving transport equations in non-orthogonal curvilinear coordinates it is assumed that the wall is in the $\xi^1 - \xi^2$ (or $\xi - \eta$) plane, and $\xi^3 \equiv \zeta$ is the wall-normal direction[1]. In this reference frame, only the U- and V-momentum equations are solved across the subgrid and only diffusion terms involving gradients in the ζ-direction are significant.

D.1 Convection of Momentum

In Appendix C, the convection term in the momentum equation was derived as follows:

$$(\mathbf{U} \cdot \nabla)\, \mathbf{U} = \left(\frac{U^{(j)}}{\sqrt{g_{jj}}} \frac{\partial U^{(i)}}{\partial \xi^j} - U^{(i)} U^{(j)} \frac{\Gamma^m_{ij} g_{im}}{g_{ii}\sqrt{g_{jj}}} + U^{(j)} U^{(m)} \frac{\Gamma^i_{mj}\sqrt{g_{ii}}}{\sqrt{g_{jj} g_{mm}}} \right) \mathbf{g}_{(i)} \qquad \text{(D.1)}$$

[1]In fact, if a skewed grid is employed, the ζ-direction may be at an angle to the wall other than 90^o.

where $U^{(i)}$ represents the physical velocity components aligned to the curvilinear base vectors and the last two terms, involving Christoffel symbols, arise from the use of a non-uniform grid. The UMIST-N wall function employs an upwind scheme for discretizing the convection term. However, rather than use the above expression for convection of momentum, the UMIST-N wall function transforms the velocity components in the upstream cell from the coordinate system used in the upstream cell into the coordinate system used in the current cell. To illustrate this, Figure D.1 shows a curved surface with a wall-parallel U-velocity in the positive ξ-direction and W-velocity in the ζ-direction. Using an upwind scheme, the wall-parallel convection term for node P is calculated by transforming the upstream velocity at node W from the coordinate frame used in cell W into the coordinate frame of cell P. Since the velocity components in the upstream and the current cells are expressed using identical base vectors (i.e. base vectors do not rotate between the adjacent cells), gradients of the metric tensors are zero. This means that the Christoffel symbols appearing in Equation D.1 are zero and the expression simplifies to:

$$(\mathbf{U} \cdot \nabla)\,\mathbf{U} = \frac{U^{(j)}}{\sqrt{g_{jj}}} \left(\frac{\partial U^{(i)}}{\partial \xi^j} \right)^* \mathbf{g}_{(i)} \tag{D.2}$$

The asterisk is introduced around the velocity gradient term to denote that upstream values of $U^{(i)}$ are transformed into the coordinate system used in the current cell. In the current version of the UMIST-N wall function, the upstream velocity is transformed from the upstream covariant base vectors into Cartesian base vectors and from there into the current cell covariant base vectors. Details of these transformations which involve the Jacobian and inverse Jacobian matrices are provided in Section E.8. Reasons for adopting this practice are discussed in Appendix G.

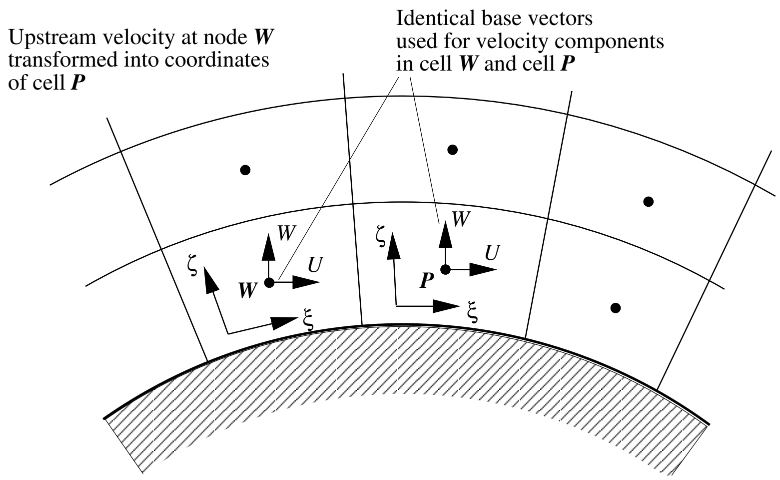

Figure D.1: Schematic of subgrid cells over a curved wall showing the transformation practice adopted for convection of momentum.

The subgrid momentum equations in non-orthogonal curvilinear coordinates can therefore be written as follows:

$$\rho\frac{\partial U^{(i)}}{\partial t}+\frac{\rho U^{(j)}}{\sqrt{g_{jj}}}\left(\frac{\partial U^{(i)}}{\partial \xi^j}\right)^{*}=\frac{1}{J}\frac{\partial}{\partial \xi^j}\left(\sqrt{g_{ii}}J\tau^{ij}\right)+S_U^i \tag{D.3}$$

where the source term, S_U^i, is given by:

$$S_U^i=-g^{ij}\sqrt{g_{ii}}\frac{\partial P'}{\partial \xi^j}-\tau^{ij}\frac{\Gamma_{ij}^m g_{im}}{\sqrt{g_{ii}}}+\tau^{mj}\Gamma_{mj}^i\sqrt{g_{ii}} \tag{D.4}$$

There is no summation on the i index in the above equations, the effective pressure is given by $P'=P+2\rho k/3$ and the stress, τ^{ij}, is expanded in Section C.5.2.

D.2 U-Momentum

It is assumed that only the wall-normal (or ζ-direction) gradient of the stress tensor is significant across the subgrid. Writing convection in the form described above, the subgrid momentum equation becomes:

$$\rho\frac{\partial U^{(i)}}{\partial t}+\frac{\rho U^{(j)}}{\sqrt{g_{jj}}}\left(\frac{\partial U^{(i)}}{\partial \xi^j}\right)^{*}=\frac{1}{J}\frac{\partial}{\partial \zeta}\left(\sqrt{g_{ii}}J\tau^{i3}\right)+S_U^i \tag{D.5}$$

where the source term, S_U^i, is given by Equation (D.4). The subgrid stress tensor, τ^{i3} is obtained from Section C.5.2, where it is assumed that only the wall-normal gradient of the wall-parallel velocity is significant:

$$\tau^{i3} \quad = \quad \mu g^{33}U_{,3}^i-\rho\overline{u^i w} \tag{D.6}$$

Using a linear eddy-viscosity turbulence model, the Reynolds stress is given by:

$$-\rho\overline{u^i w}=\mu_t g^{33}U_{,3}^i-\frac{2}{3}g^{i3}\rho k \tag{D.7}$$

and the U-momentum diffusion term can then be written, using Equation (C.99):

$$\begin{aligned}
&\frac{1}{J}\frac{\partial}{\partial \zeta}\left(\sqrt{g_{ii}}J\tau^{i3}\right)\\
&=\frac{1}{J}\frac{\partial}{\partial \zeta}\left(J\sqrt{g_{11}}\mu_{eff}g^{33}\left[\underbrace{\frac{\partial}{\partial \zeta}\left(\frac{U}{\sqrt{g_{11}}}\right)}+\frac{U^{(m)}}{\sqrt{g_{mm}}}\Gamma_{m3}^1\right]\right)\\
&-\frac{1}{J}\frac{\partial}{\partial \zeta}\left(\frac{2}{3}\rho k g^{13}J\sqrt{g_{11}}\right)
\end{aligned} \tag{D.8}$$

where ($\mu_{eff} = \mu + \mu_t$). Expanding the underbraced term using the quotient rule:

$$\begin{aligned} &\frac{1}{J}\frac{\partial}{\partial\zeta}\left[J\sqrt{g_{11}}\mu_{eff}g^{33}\frac{\partial}{\partial\zeta}\left(\frac{U}{\sqrt{g_{11}}}\right)\right] \\ &= \frac{1}{J}\frac{\partial}{\partial\zeta}\left(J\mu_{eff}g^{33}\frac{\partial U}{\partial\zeta} - \frac{J}{\sqrt{g_{11}}}\mu_{eff}g^{33}U\frac{\partial\sqrt{g_{11}}}{\partial\zeta}\right) \end{aligned} \tag{D.9}$$

where, from Equation (C.77):

$$\frac{\partial\sqrt{g_{11}}}{\partial\zeta} = \frac{g_{1m}}{\sqrt{g_{11}}}\Gamma^m_{13} \tag{D.10}$$

The diffusion term can, therefore, be written:

$$\frac{1}{J}\frac{\partial}{\partial\zeta}\left(J\mu_{eff}g^{33}\frac{\partial U}{\partial\zeta}\right) + S \tag{D.11}$$

where:

$$\begin{aligned} S &= \frac{1}{J}\frac{\partial}{\partial\zeta}\left[J\mu_{eff}g^{33}\left(U^{(m)}\frac{\sqrt{g_{11}}}{\sqrt{g_{mm}}}\Gamma^1_{m3} - U\frac{g_{1m}}{g_{11}}\Gamma^m_{13}\right)\right] \\ &\quad -\frac{1}{J}\frac{\partial}{\partial\zeta}\left(\frac{2}{3}\rho k g^{13}J\sqrt{g_{11}}\right) \end{aligned} \tag{D.12}$$

Finally, the subgrid U-momentum equation can be written:

$$\boxed{\frac{\rho U}{\sqrt{g_{11}}}\left(\frac{\partial U}{\partial\xi}\right)^* + \frac{\rho V}{\sqrt{g_{22}}}\left(\frac{\partial U}{\partial\eta}\right)^* + \frac{\rho W}{\sqrt{g_{33}}}\left(\frac{\partial U}{\partial\zeta}\right)^* = \frac{1}{J}\frac{\partial}{\partial\zeta}\left(J\mu_{eff}g^{33}\frac{\partial U}{\partial\zeta}\right) + S^1_U} \tag{D.13}$$

where the source term, S^1_U, is now given by:

$$\begin{aligned} S^1_U &= -g^{1j}\sqrt{g_{11}}\frac{\partial P'}{\partial\xi^j} - \tau^{1j}\frac{\Gamma^m_{1j}g_{1m}}{\sqrt{g_{11}}} + \tau^{mj}\Gamma^1_{mj}\sqrt{g_{11}} \\ &\quad +\frac{1}{J}\frac{\partial}{\partial\zeta}\left[J\mu_{eff}g^{33}\left(U^{(m)}\frac{\sqrt{g_{11}}}{\sqrt{g_{mm}}}\Gamma^1_{m3} - U\frac{g_{1m}}{g_{11}}\Gamma^m_{13}\right)\right] \end{aligned} \tag{D.14}$$

and the pressure term includes the isotropic component of the Reynolds stress $\left(P' = P + 2\rho k/3\right)$.

D.3 V-Momentum

The process described above to derive the subgrid U-momentum equation can be repeated to obtain the wall-parallel V-momentum equation:

$$\boxed{\frac{\rho U}{\sqrt{g_{11}}}\left(\frac{\partial V}{\partial\xi}\right)^* + \frac{\rho V}{\sqrt{g_{22}}}\left(\frac{\partial V}{\partial\eta}\right)^* + \frac{\rho W}{\sqrt{g_{33}}}\left(\frac{\partial V}{\partial\zeta}\right)^* = \frac{1}{J}\frac{\partial}{\partial\zeta}\left(J\mu_{eff}g^{33}\frac{\partial V}{\partial\zeta}\right) + S^2_U} \tag{D.15}$$

where:

$$
\begin{aligned}
S_U^2 &= -g^{2j}\sqrt{g_{22}}\frac{\partial P'}{\partial \xi^j} - \tau^{2j}\frac{\Gamma_{2j}^m g_{2m}}{\sqrt{g_{22}}} + \tau^{mj}\Gamma_{mj}^2\sqrt{g_{22}} \\
&\quad + \frac{1}{J}\frac{\partial}{\partial \zeta}\left[J\mu_{eff}g^{33}\left(U^{(m)}\frac{\sqrt{g_{22}}}{\sqrt{g_{mm}}}\Gamma_{m3}^2 - V\frac{g_{2m}}{g_{22}}\Gamma_{23}^m\right)\right] \qquad \text{(D.16)}
\end{aligned}
$$

D.4 Scalar, ϕ

The subgrid steady-flow scalar equation is obtained from Equation (C.71) and (C.130) by neglecting diffusive fluxes parallel to the wall:

$$
\boxed{\frac{\rho U}{\sqrt{g_{11}}}\frac{\partial \phi}{\partial \xi} + \frac{\rho V}{\sqrt{g_{22}}}\frac{\partial \phi}{\partial \eta} + \frac{\rho W}{\sqrt{g_{33}}}\frac{\partial \phi}{\partial \zeta} = \frac{1}{J}\frac{\partial}{\partial \zeta}\left(J J_\phi^3\right) + S_\phi} \qquad \text{(D.17)}
$$

where the scalar flux, J_ϕ^3, is given by:

$$
J_\phi^3 = \left(\Gamma_\phi + \frac{\mu_t}{\sigma_\phi}\right) g^{33}\frac{\partial \phi}{\partial \zeta} \qquad \text{(D.18)}
$$

D.5 Turbulent Kinetic Energy, k

The subgrid k-equation, obtained in a similar manner to that described above for the scalar equation, is given by:

$$
\boxed{\frac{\rho U}{\sqrt{g_{11}}}\frac{\partial k}{\partial \xi} + \frac{\rho V}{\sqrt{g_{22}}}\frac{\partial k}{\partial \eta} + \frac{\rho W}{\sqrt{g_{33}}}\frac{\partial k}{\partial \zeta} = \frac{1}{J}\frac{\partial}{\partial \zeta}\left[J g^{33}\left(\mu + \frac{\mu_t}{\sigma_k}\right)\frac{\partial k}{\partial \zeta}\right] + G - \rho\varepsilon} \qquad \text{(D.19)}
$$

where the production rate of turbulent kinetic energy, G, is as follows:

$$
G = -\rho g_{im}\overline{u^j u^m}U^i_{,j} \qquad \text{(D.20)}
$$

All the components of the production term are included in the UMIST-N wall function to account for turbulence generation due to both normal and shear stress. Expressions for the Reynolds stress $\left(-\rho\overline{u^i u^j}\right)$ and strain-rate $\left(U^i_{,j}\right)$ are given above (see Equations C.89 to C.105). The expanded production term, G, has three dummy indices and therefore expands to 27 terms, as follows:

$$
\begin{aligned}
G \;=\; & -\rho\overline{uu}\left(g_{11}U_{,1} + g_{21}V_{,1} + g_{31}W_{,1}\right) \\
& -\rho\overline{uv}\left(g_{12}U_{,1} + g_{22}V_{,1} + g_{32}W_{,1}\right) \\
& -\rho\overline{uw}\left(g_{13}U_{,1} + g_{23}V_{,1} + g_{33}W_{,1}\right) \\
& \\
& -\rho\overline{uv}\left(g_{11}U_{,2} + g_{21}V_{,2} + g_{31}W_{,2}\right) \\
& -\rho\overline{vv}\left(g_{12}U_{,2} + g_{22}V_{,2} + g_{32}W_{,2}\right) \\
& -\rho\overline{vw}\left(g_{13}U_{,2} + g_{23}V_{,2} + g_{33}W_{,2}\right) \\
& \\
& -\rho\overline{uw}\left(g_{11}U_{,3} + g_{21}V_{,3} + g_{31}W_{,3}\right) \\
& -\rho\overline{vw}\left(g_{12}U_{,3} + g_{22}V_{,3} + g_{32}W_{,3}\right) \\
& -\rho\overline{ww}\left(g_{13}U_{,3} + g_{23}V_{,3} + g_{33}W_{,3}\right)
\end{aligned}
\tag{D.21}
$$

The expression for the total dissipation rate is simplified by only considering the gradient of $k^{1/2}$ normal to the wall (i.e. in the ζ-direction):

$$
\begin{aligned}
\varepsilon \;&=\; \tilde{\varepsilon} + 2\nu g^{jm}\left(\frac{\partial k^{1/2}}{\partial \xi^m}\right)\left(\frac{\partial k^{1/2}}{\partial \xi^j}\right) \\
&\approx\; \tilde{\varepsilon} + 2\nu g^{33}\left(\frac{\partial k^{1/2}}{\partial \zeta}\right)\left(\frac{\partial k^{1/2}}{\partial \zeta}\right)
\end{aligned}
\tag{D.22}
$$

D.6 Dissipation Rate, $\tilde{\varepsilon}$

The subgrid $\tilde{\varepsilon}$-equation is given by:

$$
\boxed{
\begin{aligned}
\frac{\rho U}{\sqrt{g_{11}}}\frac{\partial \tilde{\varepsilon}}{\partial \xi} + \frac{\rho V}{\sqrt{g_{22}}}\frac{\partial \tilde{\varepsilon}}{\partial \eta} + \frac{\rho W}{\sqrt{g_{33}}}\frac{\partial \tilde{\varepsilon}}{\partial \zeta} = \frac{1}{J}\frac{\partial}{\partial \zeta}\left[J g^{33}\left(\mu + \frac{\mu_t}{\sigma_\varepsilon}\right)\frac{\partial \tilde{\varepsilon}}{\partial \zeta}\right] \\
+ c_{\varepsilon 1} f_1 G \frac{\tilde{\varepsilon}}{k} - c_{\varepsilon 2} f_2 \rho \frac{\tilde{\varepsilon}^2}{k} + \rho Y_c + P_{\varepsilon 3}
\end{aligned}
}
\tag{D.23}
$$

The source terms appearing in the $\tilde{\varepsilon}$-equation include production $(c_{\varepsilon 1} G\tilde{\varepsilon}/k)$, dissipation $\left(c_{\varepsilon 2}\tilde{\varepsilon}^2/k\right)$, Yap correction (Y_c) and the near-wall gradient production source term $(P_{\varepsilon 3})$. The production term, G, is expanded as above, Equation (D.21). The full expansion of the gradient production source term, $P_{\varepsilon 3}$, is given by:

$$
P_{\varepsilon 3} = 2\mu\nu_t g_{im} g_{jn} g^{lp}\left(g^{no}U^m_{,o}\right)_{,p}\left(g^{jk}U^i_{,k}\right)_{,l}
\tag{D.24}
$$

where, from Equation (B.109), the double-derivative of the velocity component is given by:

$$\left(g^{jk}U^i_{,k}\right)_{,l} = \frac{\partial\left(g^{jk}U^i_{,k}\right)}{\partial\xi^l} + g^{jk}U^m_{,k}\Gamma^i_{ml} + g^{mk}U^i_{,k}\Gamma^j_{ml} \tag{D.25}$$

Clearly, the full expansion of $P_{\varepsilon 3}$ cannot be used within the UMIST-N wall function without considerable cost. To simplify this term, it is assumed that only the gradient of the wall-parallel U and V velocity components in the wall-normal ζ-direction are significant (i.e. $k = l = o = p = 3$ and $i = m = 1, 2$):

$$\begin{aligned}
P_{\varepsilon 3} &= 2\mu\nu_t\left[g_{11}g_{jn}g^{33}\left(g^{n3}U_{,3}\right)_{,3}\left(g^{j3}U_{,3}\right)_{,3} + g_{22}g_{jn}g^{33}\left(g^{n3}V_{,3}\right)_{,3}\left(g^{j3}V_{,3}\right)_{,3}\right] && \text{(D.26)}\\
&= 2\mu\nu_t g_{11}g^{33}. \\
&\quad \left\{g_{11}\left[\left(g^{13}U_{,3}\right)_{,3}\right]^2 + g_{22}\left[\left(g^{23}U_{,3}\right)_{,3}\right]^2 + g_{33}\left[\left(g^{33}U_{,3}\right)_{,3}\right]^2\right. \\
&\quad \left.+2g_{12}\left(g^{13}U_{,3}\right)_{,3}\left(g^{23}U_{,3}\right)_{,3} + 2g_{13}\left(g^{13}U_{,3}\right)_{,3}\left(g^{33}U_{,3}\right)_{,3} + 2g_{23}\left(g^{23}U_{,3}\right)_{,3}\left(g^{33}U_{,3}\right)_{,3}\right\} \\
&\quad +2\mu\nu_t g_{22}g^{33}. \\
&\quad \left\{g_{11}\left[\left(g^{13}V_{,3}\right)_{,3}\right]^2 + g_{22}\left[\left(g^{23}V_{,3}\right)_{,3}\right]^2 + g_{33}\left[\left(g^{33}V_{,3}\right)_{,3}\right]^2\right. \\
&\quad \left.+2g_{12}\left(g^{13}V_{,3}\right)_{,3}\left(g^{23}V_{,3}\right)_{,3} + 2g_{13}\left(g^{13}V_{,3}\right)_{,3}\left(g^{33}V_{,3}\right)_{,3} + 2g_{23}\left(g^{23}V_{,3}\right)_{,3}\left(g^{33}V_{,3}\right)_{,3}\right\} && \text{(D.27)}
\end{aligned}$$

where the double-derivative term is obtained from:

$$\left(g^{j3}U^i_{,3}\right)_{,3} = \frac{\partial\left(g^{j3}U^i_{,3}\right)}{\partial\zeta} + g^{j3}\left(U_{,3}\Gamma^i_{13} + V_{,3}\Gamma^i_{23}\right) + U^i_{,3}\left(g^{13}\Gamma^j_{13} + g^{23}\Gamma^j_{23} + g^{33}\Gamma^j_{33}\right) \tag{D.28}$$

The above expressions require the cell boundary values of strain-rates $U_{,3}$ and $V_{,3}$ (given by Equations C.99 and C.102), the contravariant metric tensor g^{33} and the Jacobian, J.

D.7 Non-Linear EVM

In the non-linear eddy-viscosity model (NLEVM) of Craft *et al.* [30], the Reynolds stress is a function of linear, quadratic and cubic combinations of strain-rate and vorticity. The constitutive equation for the Reynolds stress anisotropy, a^{ij}, defined as the traceless ratio of the Reynolds stress to the turbulent kinetic energy is given in Equation (2.27) for Cartesian coordinates. In order to satisfy the summation convention in non-orthogonal curvilinear coordinates (i.e. summation between repeated upper and lower pairs of indices) this becomes:

$$\begin{aligned}
a^{ij} &= \frac{\overline{u^i u^j}}{k} - \frac{2}{3} g^{ij} \\
&= -\frac{\nu_t}{k} S^{ij} \\
&\quad + c_1 \frac{\nu_t}{\tilde{\varepsilon}} \left(g_{kl} S^{ik} S^{jl} - \frac{1}{3} g_{km} g_{ln} S^{kl} S^{mn} g^{ij} \right) \\
&\quad + c_2 \frac{\nu_t}{\tilde{\varepsilon}} \left(g_{kl} \Omega^{ik} S^{jl} + g_{kl} \Omega^{jk} S^{il} \right) \\
&\quad + c_3 \frac{\nu_t}{\tilde{\varepsilon}} \left(g_{kl} \Omega^{ik} \Omega^{jl} - \frac{1}{3} g_{km} g_{ln} \Omega^{kl} \Omega^{mn} g^{ij} \right) \\
&\quad + c_4 \frac{\nu_t k}{\tilde{\varepsilon}^2} \left(S^{ki} \Omega^{lj} + S^{kj} \Omega^{li} \right) g_{lm} g_{kn} S^{mn} \\
&\quad + c_5 \frac{\nu_t k}{\tilde{\varepsilon}^2} \left(g_{kl} g_{mn} \Omega^{ik} \Omega^{ln} S^{mj} + g_{kl} g_{mn} S^{ik} \Omega^{ln} \Omega^{mj} - \frac{2}{3} g_{kl} g_{mo} g_{np} S^{km} \Omega^{on} \Omega^{pl} g^{ij} \right) \\
&\quad + c_6 \frac{\nu_t k}{\tilde{\varepsilon}^2} g_{km} g_{ln} S^{ij} S^{ml} S^{kn} \\
&\quad + c_7 \frac{\nu_t k}{\tilde{\varepsilon}^2} g_{km} g_{ln} S^{ij} \Omega^{ml} \Omega^{kn}
\end{aligned} \tag{D.29}$$

with non-orthogonal curvilinear strain-rate and vorticity tensors:

$$S^{ij} = g^{jm} U^i_{,m} + g^{im} U^j_{,m} \qquad \Omega^{ij} = g^{jm} U^i_{,m} - g^{im} U^j_{,m} \tag{D.30}$$

The Craft *et al.* model also involves a c_μ function which is sensitized to the dimensionless strain-rate and vorticity invariants. In Cartesian coordinates these are given by:

$$\tilde{S} = \frac{k}{\tilde{\varepsilon}} \sqrt{\frac{1}{2} S_{ij} S_{ij}} \qquad \tilde{\Omega} = \frac{k}{\tilde{\varepsilon}} \sqrt{\frac{1}{2} \Omega_{ij} \Omega_{ij}} \tag{D.31}$$

In order to satisfy the summation convention in non-orthogonal curvilinear coordinates two additional contravariant metric tensors must be introduced into the above expression:

$$\tilde{S} = \frac{k}{\tilde{\varepsilon}} \sqrt{\frac{1}{2} g_{ik} g_{jl} S^{kj} S^{il}} \qquad \tilde{\Omega} = \frac{k}{\tilde{\varepsilon}} \sqrt{\frac{1}{2} g_{ik} g_{jl} \Omega^{kj} \Omega^{il}} \tag{D.32}$$

The recent Craft *et al.* paper [67] introduced an additional term in the c_μ function involving the dimensionless third invariant of the strain-rate tensor, S_I. In Cartesian coordinates this is given by:

$$S_I = \frac{S_{ij} S_{jk} S_{ki}}{(S_{nl} S_{nl}/2)^{3/2}} \tag{D.33}$$

and in curvilinear coordinates:

$$S_I = \frac{g_{il} g_{jm} g_{kn} S^{lj} S^{mk} S^{ni}}{(g_{oq} g_{pr} S^{qp} S^{or}/2)^{3/2}} \tag{D.34}$$

The S_I term in non-orthogonal curvilinear coordinates, with ten dummy indices (i,j,k,l,m,n,o,p,q,r), is extremely expensive to calculate fully and was not used in the non-orthogonal UMIST-N wall function implementation of the Craft *et al.* model. To improve the numerical stability of the model, the tensorially linear c_6 and c_7 terms in Equation D.29 are treated as effective viscosity terms when their sum is negative (see discussion in Section 2.3). The Reynolds stress is then given by:

$$-\rho\overline{u^i u^j} = \mu_t' S^{ij} - \rho\widehat{\overline{u^i u^j}} - \frac{2}{3}g^{ij}\rho k \tag{D.35}$$

where the modified eddy-viscosity, μ_t', and the remaining higher-order components of the Reynolds stress, $\widehat{\overline{u_i u_j}}$, are given by:

$$\mu_t' = \mu_t - \mu_t\frac{k^2}{\tilde{\varepsilon}^2}\min\left[\left(c_6 g_{km}g_{ln}S^{ml}S^{kn} + c_7 g_{km}g_{ln}\Omega^{ml}\Omega^{kn}\right), 0\right] \tag{D.36}$$

$$\begin{aligned}
\widehat{\overline{u^i u^j}} \;=\; & c_1\frac{\nu_t k}{\tilde{\varepsilon}}\left(g_{kl}S^{ik}S^{jl} - \frac{1}{3}g_{km}g_{ln}S^{kl}S^{mn}g^{ij}\right)\\
& + c_2\frac{\nu_t k}{\tilde{\varepsilon}}\left(g_{kl}\Omega^{ik}S^{jl} + g_{kl}\Omega^{jk}S^{il}\right)\\
& + c_3\frac{\nu_t k}{\tilde{\varepsilon}}\left(g_{kl}\Omega^{ik}\Omega^{jl} - \frac{1}{3}g_{km}g_{ln}\Omega^{kl}\Omega^{mn}g^{ij}\right)\\
& + c_4\frac{\nu_t k^2}{\tilde{\varepsilon}^2}\left(S^{ki}\Omega^{lj} + S^{kj}\Omega^{li}\right)g_{lm}g_{kn}S^{mn}\\
& + c_5\frac{\nu_t k^2}{\tilde{\varepsilon}^2}\left(g_{kl}g_{mn}\Omega^{ik}\Omega^{ln}S^{mj} + g_{kl}g_{mn}S^{ik}\Omega^{ln}\Omega^{mj} - \frac{2}{3}g_{kl}g_{mo}g_{np}S^{km}\Omega^{on}\Omega^{pl}g^{ij}\right)\\
& + \nu_t\frac{k^2}{\tilde{\varepsilon}^2}\max\left[\left(c_6 g_{km}g_{ln}S^{ml}S^{kn} + c_7 g_{km}g_{ln}\Omega^{ml}\Omega^{kn}\right), 0\right]
\end{aligned} \tag{D.37}$$

Using the NLEVM, the subgrid momentum equation becomes:

$$\boxed{\frac{\rho U}{\sqrt{g_{11}}}\left(\frac{\partial U^{(i)}}{\partial \xi}\right)^* + \frac{\rho V}{\sqrt{g_{22}}}\left(\frac{\partial U^{(i)}}{\partial \eta}\right)^* + \frac{\rho W}{\sqrt{g_{33}}}\left(\frac{\partial U^{(i)}}{\partial \zeta}\right)^* = \frac{1}{J}\frac{\partial}{\partial \zeta}\left(J\mu_{eff}' g^{33}\frac{\partial U^{(i)}}{\partial \zeta}\right) + S_U^i} \tag{D.38}$$

where $i = 1$ and $i = 2$ for the subgrid U- and V-momentum equations, respectively. The source term, S_U^i, is now given by:

$$\begin{aligned}
S_U^i \;=\; & -g^{ij}\sqrt{g_{ii}}\frac{\partial P'}{\partial \xi^j} - \tau^{ij}\frac{\Gamma_{ij}^m g_{im}}{\sqrt{g_{ii}}} + \tau^{mj}\Gamma_{mj}^i\sqrt{g_{ii}}\\
& + \frac{1}{J}\frac{\partial}{\partial \zeta}\left[J\mu_{eff}' g^{33}\left(U^{(m)}\frac{\sqrt{g_{ii}}}{\sqrt{g_{mm}}}\Gamma_{m3}^i - U\frac{g_{im}}{g_{ii}}\Gamma_{i3}^m\right)\right]\\
& \underbrace{\frac{1}{J}\frac{\partial}{\partial \zeta}\left(\sqrt{g_{ii}}J\rho\widehat{\overline{u^i w}}\right)}
\end{aligned} \tag{D.39}$$

where the underbraced term in Equation (D.39) is a new term introduced by the NLEVM, the modified eddy-viscosity is given by $\mu'_{eff} = \mu + \mu'_t$, and the stress tensor terms in the above equations are now calculated using:

$$\tau^{ij} = \mu'_{eff} S^{ij} - \rho \widehat{\overline{u^i u^j}} - \frac{2}{3} g^{ij} \rho k \tag{D.40}$$

D.8 Differential Yap Correction

The differential length-scale correction developed by Iacovides & Raisee [66] is given by:

$$Y_{dc} = c_w \frac{\tilde{\varepsilon}^2}{k} max \left[F \left(F+1 \right)^2, 0 \right] \tag{D.41}$$

where:

$$F = \frac{1}{c_l} \left[\left(\frac{\partial l}{\partial x_j} \frac{\partial l}{\partial x_j} \right)^{1/2} - dl_e dy \right] \tag{D.42}$$

The constants c_w and c_l and the function $dl_e dy$ are given in Section 2.2 and l is the turbulence length scale, calculated from $l = k^{3/2}/\varepsilon$. The gradient of the length scale in non-orthogonal curvilinear coordinates given by:

$$F = \frac{1}{c_l} \left[\left(g^{jk} \frac{\partial l}{\partial \xi^j} \frac{\partial l}{\partial \xi^k} \right)^{1/2} - dl_e dy \right] \tag{D.43}$$

In the UMIST-N wall function, this is simplified to:

$$F = \frac{1}{c_l} \left[\left(g^{33} \frac{\partial l}{\partial \zeta} \frac{\partial l}{\partial \zeta} \right)^{1/2} - dl_e dy \right] \tag{D.44}$$

where it is assumed that the gradient of the length scale parallel to the wall is negligible in comparison with the gradient normal to the wall.